1942

Statical equilibrium of skew and sector-shaped plates

Ralph H. Tripp
Iowa State College

STATICAL EQUILIBRIUM OF SKEW AND

SECTOR-SHAPED PLATES

by

Ralph H. Tripp

A Thesis Submitted to the Graduate Faculty
for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject: Applied Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State College

1942

Acknowledgement:

The writer takes this opportunity to express his sincere appreciation to

Dr. D. L. Holl

for his constant encouragement during the preparation of this thesis.

TABLE OF CONTENTS

Introduction

The study of the equilibrium of Skew and Sector-shaped Plates has received considerable attention in the last few years. A 'skew plate' is taken to mean a plate which has the shape of a parallelogram. As early as 1933, B. G. Galerkin published a book entitled 'Elastic Thin Plates'[1] in which he considered certain special skew plates as well as many kinds of sector-shaped plates with various boundary conditions. C. V. Brigatti[2] also published a solution of the problem of the skew plate but his method and results seem to be in question.[3] In 1939 Adolf Anzelius[4] published a solution for a plate of 45° skew and in 1940 Helmut Vogt[5] based a doctoral dissertation upon the solution of the problem of a skew plate with two opposite edges simply supported, and carrying a uniform load. The University of Illinois sponsored the publication of a bulletin[6] by Jensen which is devoted exclusively to the subject of skew plates. In this bulletin special emphasis is laid on the types of plates and boundary conditions which are applicable to road slabs.

The fact that highways are being straightened as much as possible to accommodate the increasing speed of modern traffic makes the analysis of skew slabs of constantly increasing importance. It is no longer desirable in most cases to put a curve in a highway in order to cross a river or a right of way at right angles, and therefore quite often the roadbed of the bridge or overpass is in the shape of a skew plate. A knowledge of where the greatest stresses and strains occur in such a plate is essential to one who is responsible for the design. The purpose of this paper is to set up a function which will yield the various stresses and strains at any point on the plate if the loading and the method of support are of a certain type.

The notation which is used throughout this paper is that used by A. Nadai in his book "Elastische Platten".[7] The fundamental assumptions which are made on the material and on the forces and displacements make this a so-called 'Thin Plate Problem'. These assumptions are:

(1) There is no stretching of the middle surface,

(2) A plane section perpendicular to the middle surface before bending remains plane after bending and normal to deflected surface,

(3) The material of the plate is isotropic,

(4) The weight of the plate itself is to be neglected,

(5) Only the effects of pure bending and torsion will be considered,

(6) The thickness of the plate is small in comparison to the lateral dimensions,

(7) The maximum deflection must not be greater than one half the thickness of the plate.

The derivation and the validity of the Lagrange Plate Equation depends directly on these assumptions which are due to Kirchoff.[8]

Chapter I

Skew Plate with a Uniform Load

(1) Solution of the Plate Equation.

The differential equation for 'Thin Plates' is known as the 'Lagrange Plate Equation'. It is written:

$$N\nabla^4 w = N\left\{\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right\} = p, \qquad (1.1)$$

where:

$$N = \frac{Eh^3}{12(1-\nu^2)},$$

p = constant normal load per unit of area,

E = Young's Modulus, (1.2)

h = thickness of plate,

ν = Poisson's ratio.

In order to obtain an exact solution for the stresses and strains of a 'Thin Plate' it is necessary to construct a deflection function, w, which will satisfy the equation (1.1) and the associated boundary conditions. Two such functions are constructed and designated by w_1, and w_2. The deflection function w_1, will be valid in the upper half of the plate where: (see Fig. 1)

$$0 \leq y \leq a,$$

and the deflection function w_2, will be valid in the lower half of the plate where: (see Fig. 1)

$$-a \leqq y \leqq 0.$$

These functions are defined as follows:

$$w_1 = f_1 + \psi_1,$$
$$w_2 = f_2 + \psi_2,$$

where:

$$f_1 = \frac{p}{96N}\left\{(x+y-a)^4 + 2a(x+y-a)^3 - a^3(x+y-a)\right\},$$
$$f_2 = \frac{p}{96N}\left\{(x+y)^4 - 2a(x+y)^3 + a^3(x+y)\right\}, \quad (1.3)$$

$$\begin{aligned}\psi_1 = \sum_{n=1,3,5\ldots}\Big\{ & A_{n_1}\left[\sinh\alpha(a-y)\sin\alpha x - \sinh\alpha x\sin\alpha y\right] + \\ & + B_{n_1}\left[\sinh\alpha(a-x)\sin\alpha y - \sinh\alpha y\sin\alpha x\right] + \\ & + C_{n_1}\left[(a-y)\cosh\alpha(a-y)\sin\alpha x - x\cosh\alpha x\sin\alpha y\right] + \\ & + D_{n_1}\left[(a-x)\cosh\alpha(a-x)\sin\alpha y - y\cosh\alpha y\sin\alpha x\right]\Big\},\end{aligned}$$

$$\begin{aligned}\psi_2 = \sum_{n=1,3,5\ldots}\Big\{ & A_{n_2}\left[\sinh\alpha y\sin\alpha x - \sinh\alpha x\sin\alpha y\right] + \\ & + B_{n_2}\left[\sinh\alpha(a-x)\sin\alpha y + \sinh\alpha(a+y)\sin\alpha x\right] + \\ & + C_{n_2}\left[y\cosh\alpha y\sin\alpha x - x\cosh\alpha x\sin\alpha y\right] + \\ & + D_{n_2}\left[(a-x)\cosh\alpha(a-x)\sin\alpha y + (a+y)\cosh\alpha(a+ \\ & + y)\sin\alpha x\right]\Big\},\end{aligned} \quad (1.4)$$

in which,

$$\alpha = n\pi/a .$$

They satisfy the plate equation (1.1)

since

$$\nabla^4 f_1 = \nabla^4 f_2 = p/N,$$
$$\nabla^4\psi_1 = \nabla^4\psi_2 = 0. \quad (1.5)$$

(2) Boundary Conditions.

Since the plate pictured in Figure 1 is to be considered as simply supported, or in other words to have pinned edges, there are eight exterior boundary conditions which the deflection functions must satisfy. The deflection and the normal bending moment must vanish identically at all points on the boundary of the plate.

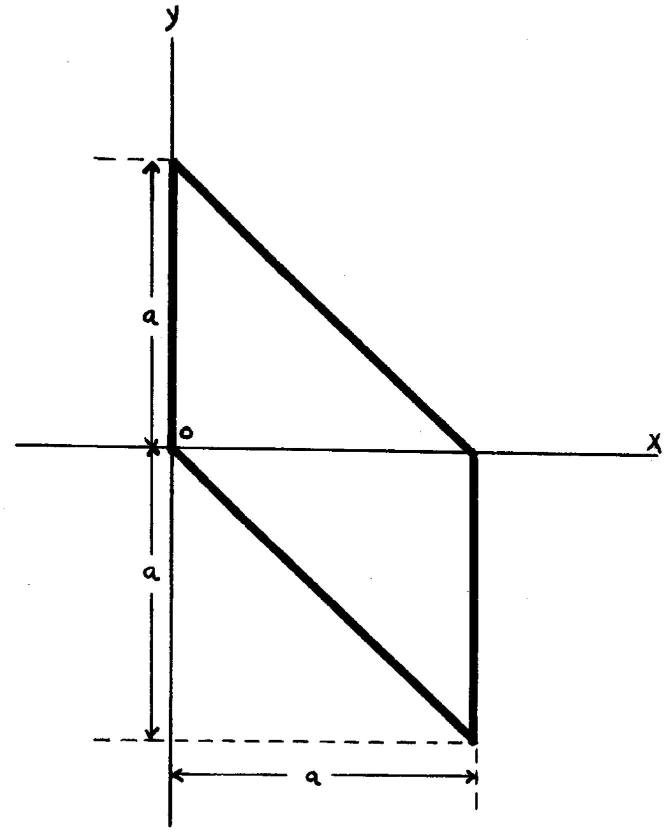


Fig. 1

In mathematical terms these conditions may be written:

$$
\begin{aligned}
&w_1\Big|_{x+y=a} = w_1\Big|_{x=0} = w_2\Big|_{x+y=0} = w_2\Big|_{x=a} = 0, \\
&M_x(w_1)\Big|_{x=0} = M_x(w_2)\Big|_{x=a} = 0, \qquad\qquad (2.1) \\
&M_n(w_1)\Big|_{x+y=a} = M_n(w_2)\Big|_{x+y=0} = 0.
\end{aligned}
$$

It is easy to demonstrate by direct substitution that the deflection functions w_1 and w_2

automatically satisfy all of the boundary conditions on the two edges, $x + y = a$ and $x + y = 0$. It should be noted that the condition which involves the normal bending moment on the skew edges is equivalent to the condition that the Laplacean in terms of the normal and tangential directions should vanish, that is:

$$\frac{\partial^2 w}{\partial n^2} + \frac{\partial^2 w}{\partial t^2} = 0.$$

This conclusion follows quite readily since there is no deflection on a pinned edge, thus making the tangential curvature zero and leaving only the condition that

$$\frac{\partial^2 w}{\partial n^2} = 0.$$

Four more exterior boundary conditions must still be satisfied by the evaluation of constants.

In addition to the exterior boundary conditions, there are four interior conditions to be met. It is necessary that the deflection functions w_1 and w_2 be matched across the short diagonal, $y = 0$, in such a way that the deflection, slope, moment, and shear are identically equal for all values of x along the line where $y = 0$. In mathematical terms these conditions may be written:

Deflection:

$$w_1\Big|_{y=0} \underset{x}{\equiv} w_2\Big|_{y=0}, \qquad (2.2)$$

Slope:

$$\frac{\partial w_1}{\partial y} \underset{x}{\equiv} \frac{\partial w_2}{\partial y}\ , \text{ at } y = 0, \tag{2.3}$$

Moment:

$$\frac{\partial^2 w_1}{\partial y^2} + \nu \frac{\partial^2 w_1}{\partial x^2} \underset{x}{\equiv} \frac{\partial^2 w_2}{\partial y^2} + \nu \frac{\partial^2 w_2}{\partial x^2}\ , \text{ at } y = 0, \tag{2.4}$$

Shear:

$$\frac{\partial}{\partial y} \nabla^2 w_1 \underset{x}{\equiv} \frac{\partial}{\partial y} \nabla^2 w_2\ , \text{ at } y = 0. \tag{2.5}$$

It must follow that if these two functions, w_1 and w_2, are matched identically for all values of x at $y = 0$, (2.2), then certainly derivatives of these functions with respect to x at $y = 0$ must be identically equal also. Specifically

$$\frac{\partial^2 w_1}{\partial x^2} = \frac{\partial^2 w_2}{\partial x^2}\ , \text{ at } y = 0,$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 w_1}{\partial x^2}\right) = \frac{\partial}{\partial y}\left(\frac{\partial^2 w_1}{\partial x^2}\right)\ , \text{ at } y = 0,$$

whence (2.4) and (2.5) may be simplified considerably since they are equivalent to the following:

$$\frac{\partial^2 w_1}{\partial y^2} = \frac{\partial^2 w_2}{\partial y^2}\ , \text{ at } y = 0, \tag{2.4'}$$

$$\frac{\partial^3 w_1}{\partial y^3} = \frac{\partial^3 w_2}{\partial y^3}\ , \text{ at } y = 0. \tag{2.5'}$$

Now we have eight arbitrary constants in the functions (1.4) and we have eight conditions to be satisfied. Accordingly the problem resolves itself into the evaluation of the arbitrary constants.

(3) Evaluation of Constants.

The boundary conditions at the edge $x = 0$ are:

$$w_1 = 0, \text{ at } x = 0, \tag{3.1}$$

$$M_x = -N\left(\frac{\partial^2 w_1}{\partial x^2} + \nu\frac{\partial^2 w_1}{\partial y^2}\right) = -N\frac{\partial^2 w_1}{\partial x^2} = 0, \text{ at } x = 0,$$

whence:

$$B_{n_1}\sinh n\pi \sin \alpha y + D_{n_1} a\cosh n\pi \sin \alpha y + f_1\Big|_{x=0} \underset{y}{\equiv} 0, \tag{3.2}$$

and

$$B_{n_1}\alpha^2\sinh n\pi \sin \alpha y + D_{n_1}\alpha^2 a\cosh n\pi \sin \alpha y +$$

$$+ D_{n_1} 2\alpha\sinh n\pi \sin \alpha y + \frac{\partial^2 f_1}{\partial x^2}\Big|_{x=0} \underset{y}{\equiv} 0. \tag{3.3}$$

The terms, f_1 and $\partial^2 f_1/\partial x^2$, may be expanded into sine series and this permits us to write (3.2) and (3.3) in the following form:

$$B_{n_1} + D_{n_1} a\coth n\pi + \frac{Pa^4 \operatorname{csch} n\pi}{N\pi^5 n^5} = 0, \tag{3.4}$$

$$B_{n_1} + D_{n_1}\left(a\coth n\pi + \frac{2}{\alpha}\right) - \frac{Pa^4 \operatorname{csch} n\pi}{N\pi^5 n^5} = 0.$$

The solution of these equations (3.4) for B_{n_1} and D_{n_1} yields:

$$B_{n_1} = -\frac{Pa^4 \operatorname{csch} n\pi}{N\pi^4 n^4}(\coth n\pi + 1/n\pi),$$

$$D_{n_1} = \frac{Pa^3 \operatorname{csch} n\pi}{N\pi^4 n^4}. \tag{3.5}$$

The boundary conditions at the edge $x = a$ are:

$$w_2 = 0, \text{ at } x = a, \tag{3.6}$$

$$M_x = -N\left\{\frac{\partial^2 w_2}{\partial x^2} + \nu \frac{\partial^2 w_2}{\partial y^2}\right\} = -N\frac{\partial^2 w_2}{\partial x^2} = 0, \text{ at } x = a.$$

These equations (3.6) may be solved in a manner very similar to the way in which we solved the preceeding equations, (3.1), and the result is:

$$B_{n_1} = A_{n_2},$$

$$D_{n_1} = C_{n_2}. \tag{3.7}$$

All the exterior boundary conditions are satisfied with the evaluation of these four constants (3.7) and (3.5). It remains to match the two deflection functions at the line where $y = 0$. The two functions, f_1 and f_2 will be seen to be identical for all values of x at the line $y = 0$ so it will only be necessary to impose the conditions (2.2), (2.3), (2.4), and (2.5) on the functions, ψ_1 and ψ_2.

From (2.2) one may obtain the following equation:

$$\sum_{n=1,3,5\cdots}\left\{A_{n_1}\sinh n\pi \sin \alpha x + C_{n_1} a\cosh n\pi \sin \alpha x\right\} \underset{x}{\equiv} \underset{x}{\equiv} \sum_{n=1,3,5\cdots}\left\{B_{n_2}\sinh n\pi \sin \alpha x + D_{n_2} a\cosh n\pi \sin \alpha x\right\} . \tag{3.8}$$

This may be written:

$$\sum_{n=1,3,5\cdots}\left\{(A_{n_1} - B_{n_2})\sinh n\pi + (C_{n_1} - D_{n_2}) a\cosh n\pi\right\}\sin \alpha x \underset{x}{\equiv} 0,$$

where in order to satisfy the identity it is necessary that:

$$A_{n_1} - B_{n_2} + (C_{n_1} - D_{n_2}) a\coth n\pi = 0. \tag{3.9}$$

From (2.4) one may obtain the following equation:

$$\sum_{n=1,3,5\cdots}\left\{ A_{n_1}\alpha^2\sinh n\pi \sin \alpha x + C_{n_1}(a\alpha^2\cosh n\pi + + 2\alpha\sinh n\pi)\sin \alpha x\right\} \underset{x}{\equiv} \sum_{n=1,3,5\cdots}\left\{B_{n_2}\alpha^2\sinh n\pi \sin \alpha x + + D_{n_2}(a\alpha^2\cosh n\pi + 2\alpha\sinh n\pi)\sin \alpha x\right\} . \tag{3.10}$$

This may be written:

$$\sum_{n=1,3,5\cdots}\left\{\left[(A_{n_1}\alpha^2\sinh n\pi + C_{n_1}\alpha^2 a\cosh n\pi) + + 2C_{n_1}\alpha\sinh n\pi - (B_{n_2}\alpha^2\sinh n\pi + D_{n_2} a\alpha^2\cosh n\pi) + + 2D_{n_2}\alpha\sinh n\pi\right] \sin \alpha x\right\} \underset{x}{\equiv} 0,$$

or simply

$$A_{n_1} - B_{n_2} + (C_{n_1} - D_{n_2})\ (\alpha\coth n\pi + 2/\alpha) = 0. \qquad (3.11)$$

The solution of (3.9) and (3.11) gives the following relations:

$$\begin{aligned} A_{n_1} &= B_{n_2}, \\ C_{n_1} &= D_{n_2}. \end{aligned} \qquad (3.12)$$

(4) Systems of Infinite Equations and their Solution.

It will be expedient at this point to change the notation slightly and let,

$$\begin{aligned} A_{n_1} &= A_n, \\ B_{n_1} &= B_n, \\ C_{n_1} &= C_n, \\ D_{n_1} &= D_n. \end{aligned} \qquad (4.1)$$

Substitution of (1.4) into the condition on the shear (2.5) results in the equation:

$$\sum_{n=1,3,5\ldots}\Big\{C_n\big[-2\alpha^2\cosh n\pi\sin\alpha x - 2\alpha^2\sinh\alpha x\big] + D_n\big[2\alpha^2\sinh\alpha(a-x) - 2\alpha^2\sin\alpha x\big]\Big\} \underset{x}{\equiv} \sum_{n=1,3,5\ldots}\Big\{D_n\big[2\alpha^2\sin\alpha x - 2\alpha^2\sinh\alpha x\big] + C_n\big[2\alpha^2\sinh\alpha(a-x) + 2\alpha^2\cosh n\pi\sin\alpha x\big]\Big\},$$

or

$$\sum_{n=1,3,5\ldots}\Big\{2\alpha^2\big[C_n\cosh n\pi + D_n\big]\sin\alpha x + \alpha^2(C_n - D_n)\big[\sinh\alpha x + \sinh\alpha(a-x)\big]\Big\} \underset{x}{\equiv} 0. \qquad (4.2)$$

When the latter part of this equation (4.2) is expanded into a sine series one may write:

$$\sum_{n=1,3,5\cdots}\left\{ n^2(C_n\cosh n\pi + D_n)\sin \alpha x + \right. \tag{4.3}$$

$$\left. + n^2(C_n - D_n)\sinh n\pi \sum_{r=1,3,5\cdots} \frac{2\beta\sin \beta x}{a(\alpha^2 + \beta^2)}\right\} \underset{x}{\equiv} 0,$$

where $\beta = r\pi/a$.

Substitute the value of D_n from (3.5) into the equation (4.3) and set,

$$K = Pa^4/N\pi^4$$

in order to obtain,

$$\sum_{n=1,3,5\cdots}\left\{ n^2\left[C_n\cosh n\pi + K/(an^4\sinh n\pi)\right] \sin \alpha x + n^2(C_n\sinh n\pi - \right.$$

$$\left. - K/an^4)\sum_{r=1,3,5\cdots} \frac{2\beta\sin \beta x}{a(\alpha^2 + \beta^2)}\right\} \underset{x}{\equiv} 0. \tag{4.4}$$

From this equation (4.4) it is quite apparent that there exists an infinite set of equations from which the constants C_n are to be evaluated. In particular the only way that equation (4.4) can be true for all values of x is that each of the coefficients of $\sin(\pi x/a)$, $\sin(3\pi x/a)$, $\sin(5\pi x/a)$, $\sin(7\pi x/a)$, ... etc. vanish identically. Thus one may proceed to write a doubly infinite system of equations as follows:

$$N_n + \sum_{i=1,3,5\cdots} A^{*}_{ni}N_i = \sum_{i=1,3,5\cdots} b_{ni}, \qquad n = 1,3,5\ldots. \tag{4.5}$$

where

$$N_n = C_n n^2 \sinh n\pi,$$

$$A'_{ni} = [2n\tanh n\pi]/[\pi(n^2 + i^2)] \ , \qquad (4.6)$$

$$b_{ni} = [2Kn\tanh n\pi]/[a\pi i^2(n^2 + i^2)] - K/[an^2\cosh n\pi] \ .$$

It can be shown that:[9]

$$\sum_{i=1,3,5\ldots} 1/[i^2(n^2 + i^2)] = \frac{\pi}{4n^3}[n\pi/2 - \tanh(n\pi/2)] \ , \qquad (4.7)$$

whence b_{ni} can be put into closed form and written:

$$b_n = \sum_{i=1,3,5\ldots} b_{ni} = \frac{K}{2an}[(\pi\tanh n\pi)/2 -$$

$$- 1/n - 1/(n\cosh n\pi)] \ . \qquad (4.8)$$

For the purpose of computation one should multiply the equations (4.5) by the factor $\pi/2n$ so that:

$$\pi N_n/2n + \sum_{i=1,3,5\ldots} (N_i \tanh n\pi)/(n^2 + i^2) = \pi b_n/2n.$$

$$n = 1,3,\ldots.$$

Evaluation of the right hand side of these equations for the first five values of n gives:

$$\pi b_1/2 = 0.37566 \ K/a,$$

$$\pi b_3/6 = 0.10799 \ K/a,$$

$$\pi b_5/10 = 0.04307 \ K/a, \qquad (4.9)$$

$$\pi b_7/14 = 0.02290 \ K/a,$$

$$\pi b_9/18 = 0.01415 \ K/a.$$

The fact can be readily demonstrated[10] that a finite square matrix in the upper left hand corner of this infinite matrix, (4.5), may be solved for a finite number of the values N_n by the method of segments. Moreover it can be shown that the constants which are to be evaluated are never greater than a bounding value which is completely determined by the coefficients in the system of equations (4.5).[11] The solution for the first five unknowns may be found from the five equations which follow:

$$\begin{aligned}
&2.06900\, N_1 + 0.10000\, N_3 + 0.03846\, N_5 + 0.02000\, N_7 + \\
&\qquad + 0.01220\, N_9 + \ldots\ldots = 0.37566\ K/a, \\
&0.10000\, N_1 + 0.57917\, N_3 + 0.02941\, N_5 + 0.01724\, N_7 + \\
&\qquad + 0.01111\, N_9 + \ldots\ldots = 0.10799\ K/a, \\
&0.03846\, N_1 + 0.02941\, N_3 + 0.33416\, N_5 + 0.01351\, N_7 + \\
&\qquad + 0.00943\, N_9 + \ldots\ldots = 0.04307\ K/a, \\
&0.02000\, N_1 + 0.01724\, N_3 + 0.01351\, N_5 + 0.23459\, N_7 + \\
&\qquad + 0.00769\, N_9 + \ldots\ldots = 0.02289\ K/a, \\
&0.01220\, N_1 + 0.01111\, N_3 + 0.00943\, N_5 + 0.00769\, N_7 + \\
&\qquad + 0.18070\, N_9 + \ldots\ldots = 0.01415\ K/a.
\end{aligned} \tag{4.10}$$

. . . .

The values for the first five unknowns from (4.10) are:

$$\begin{aligned}
N_1 &= 0.17172\ K/a, \\
N_3 &= 0.14925\ K/a, \\
N_5 &= 0.09197\ K/a, \\
N_7 &= 0.06507\ K/a, \\
N_9 &= 0.04998\ K/a,
\end{aligned} \tag{4.11}$$

whence from (4.6),

$$
\begin{aligned}
C_1 &= 1.4869 \ K/a \times 10^{-2}, \\
C_3 &= 2.676 \ K/a \times 10^{-6}, \\
C_5 &= 1.109 \ K/a \times 10^{-9}, \\
C_7 &= 7.485 \ K/a \times 10^{-13}, \\
C_9 &= 6.52 \ K/a \times 10^{-16}.
\end{aligned}
\tag{4.12}
$$

From the slope condition at $y = 0$, (2.3), one may write the equation:

$$
\begin{aligned}
&\sum_{n=1,3,5\ldots} \Big\{ A_n\big[-\alpha\cosh n\pi \sin \alpha x - \alpha\sinh \alpha x\big] + B_n\big[\alpha\sinh \alpha(a - x) - \\
&\quad - \alpha\sin \alpha x\big] + C_n\big[(-n\pi\sinh n\pi - \cosh n\pi)\sin \alpha x - \\
&\quad - \alpha x\cosh \alpha x\big] + D_n\big[\alpha(a - x)\cosh \alpha(a - x) - \sin \alpha x\big]\Big\} \underset{x}{\equiv} \\
&\underset{x}{\equiv} \sum_{n=1,3,5\ldots} \Big\{ B_n\big[\alpha\sin \alpha x - \alpha\sinh \alpha x\big] + A_n\big[\alpha\sinh \alpha(a - x) + \\
&\quad + \alpha\cosh n\pi \sin \alpha x\big] + D_n\big[\sin \alpha x - \alpha x\cosh \alpha x\big] + \\
&\quad + C_n\big[\alpha(a - x)\cosh \alpha(a - x) + n\pi \sinh n\pi \sin \alpha x + \\
&\quad + \cosh n\pi \sin \alpha x\big]\Big\} ,
\end{aligned}
\tag{4.13}
$$

or

$$
\begin{aligned}
&\sum_{n=1,3,5\ldots} \alpha\Big\{2\big[A_n\cosh n\pi + B_n + aC_n(\sinh n\pi + (\cosh n\pi)/n\pi) + \\
&\quad + D_n/\alpha\big] \sin \alpha x + (A_n - B_n)\big[\sinh \alpha x + \sinh \alpha(a - x)\big] + \\
&\quad + (C_n - D_n)\big[x\cosh \alpha x + (a - x)\cosh \alpha(a - x)\big]\Big\} \underset{x}{\equiv} 0.
\end{aligned}
\tag{4.14}
$$

Expansion of the latter part of this equation into sine series results in:

$$\sum_{n=1,3,5\cdots}\Big\{\big[\alpha A_n\cosh n\pi + \alpha B_n + C_n(n\pi\sinh n\pi + \cosh n\pi) + D_n\big]\sin\alpha x +$$

$$+ \sum_{r=1,3,5\cdots}\frac{2\alpha\beta}{a(\alpha^2+\beta^2)}\Big[\big[A_n - B_n\big]\sinh n\pi + \big[C_n - D_n\big]\big[a\cosh n\pi -$$

$$- (2\alpha\sinh n\pi)/(\alpha^2+\beta^2)\big]\Big]\sin\beta x\Big\} \underset{x}{\equiv} 0, \qquad (4.15)$$

where as before,

$$\beta = r\pi/a.$$

Once again, following the same reasoning as was used to obtain the last system of equations, (4.5), we are able to write from (4.15):

$$M_n + \sum_{i=1,3,5\cdots}\gamma_{ni}M_i = \sum_{i=1,3,5\cdots}d_{ni} + T_n, \qquad n = 1,3,5,\ldots. \qquad (4.16)$$

where

$$M_n = nA_n\sinh n\pi,$$

$$\gamma_{ni} = (2n\tanh n\pi)/[\pi(i^2+n^2)], \qquad (4.17)$$

$$d_{ni} = (2ni\tanh n\pi)/[\pi(i^2+n^2)]B_i\sinh i\pi + (D_i - C_i)\big(a\cosh i\pi -$$

$$- [2ai\sinh i\pi]/[\pi(i^2+n^2)]\big),$$

$$T_n = -\,aC_n\big[n\tanh n\pi\,\sinh n\pi + (\sinh n\pi)/\pi\big] - \tanh n\pi\big[aD_n/\pi +$$

$$+ nB_n\big].$$

The identities (4.17) may be more concisely written as follows:

$$M_n = nA_n \sinh n\pi,$$

$$\gamma_{ni} = (2n\tanh n\pi)/[\pi(i^2 + n^2)],$$

$$d_{ni} = -\frac{2n \tanh n\pi}{\pi^2(i^2 + n^2)}\Big\{K/4i^4 + K/(2i^4 + 2i^2n^2) + + aN_i[(\pi\coth i\pi)/i - 2/(i^2 + n^2)]\Big\}, \tag{4.18}$$

$$T_n = \frac{aN_n}{n}[\tanh n\pi + 1/n\pi] + K/(4n^3\sinh n\pi).$$

The last two of the quantities in (4.18) can be summed up to give:

$$d_{i1} = 0.09108\ K,$$
$$d_{i3} = 0.05535\ K,$$
$$d_{i5} = 0.03728\ K,$$
$$d_{i7} = 0.02802\ K,$$
$$d_{i9} = 0.02229\ K,$$

and (4.19)

$$T_1 = -\ 0.20409\ K,$$
$$T_3 = -\ 0.05503\ K,$$
$$T_5 = -\ 0.01956\ K,$$
$$T_7 = -\ 0.00972\ K,$$
$$T_9 = -\ 0.00575\ K.$$

If we multiply the system (4.16) by the factor $\pi/2n$ then this system is very similar to the system (4.5). Again we may say that this system has a solution which exists but it should be noted in passing that the proof that a solution of (4.16) exists depends on the fact that we are able to prove that values N_n are bounded.[11] The first five equations of (4.16) are:

$$
\begin{aligned}
2.06900\, M_1 + 0.10000\, M_3 + 0.03846\, M_5 + 0.02000\, M_7 &+ \\
+ 0.01220\, M_9 + \ldots\ldots &= -0.46365\, K, \\
0.10000\, M_1 + 0.57917\, M_3 + 0.02941\, M_5 + 0.01724\, M_7 &+ \\
+ 0.01111\, M_9 + \ldots\ldots &= -0.05779\, K, \\
0.03846\, M_1 + 0.02941\, M_3 + 0.33416\, M_5 + 0.01351\, M_7 &+ \\
+ 0.00943\, M_9 + \ldots\ldots &= -0.01786\, K, \\
0.02000\, M_1 + 0.01724\, M_3 + 0.01351\, M_5 + 0.23459\, M_7 &+ \\
+ 0.00769\, M_9 + \ldots\ldots &= -0.00847\, K, \\
0.01220\, M_1 + 0.01111\, M_3 + 0.00943\, M_5 + 0.00769\, M_7 &+ \\
+ 0.18070\, M_9 + \ldots\ldots &= -0.00489\, K.
\end{aligned}
\tag{4.20}
$$

. . . .

The solution of (4.20) is:

$$
\begin{aligned}
M_1 &= -\,0.22063\, K, \\
M_3 &= -\,0.06010\, K, \\
M_5 &= -\,0.02212\, K, \\
M_7 &= -\,0.01138\, K, \\
M_9 &= -\,0.00686\, K,
\end{aligned}
\tag{4.21}
$$

whence, from (4.18),

$$\begin{aligned} A_1 &= -\ 1.9104\ K \times 10^{-2}, \\ A_3 &= -\ 3.233\ K \times 10^{-6}, \\ A_5 &= -\ 1.335\ K \times 10^{-9}, \\ A_7 &= -\ 9.174\ K \times 10^{-13}, \\ A_9 &= -\ 7.99\ K \times 10^{-16}. \end{aligned} \tag{4.22}$$

This concludes the formal solution of the problem since all of the boundary conditions, interior and exterior, have been satisfied and all of the constants have been evaluated.

(5) Evaluation of Deflection and Twisting Moments.

If one takes the deflection function, w_1, and substitutes the constants which have been evaluated then the deflections may be calculated for any point on the plate since by symmetry the lower half of the plate must deflect exactly the same as the upper half. Accordingly the deflections were calculated along the short diagonal where $y = 0$. They are: (see graph No. 1)

$$\begin{aligned} &\text{At } x = 0.1a, \quad w = 0.000253\ Pa^4/N, \\ &\text{At } x = 0.2a, \quad w = 0.000649\ Pa^4/N, \\ &\text{At } x = 0.3a, \quad w = 0.001022\ Pa^4/N, \\ &\text{At } x = 0.4a, \quad w = 0.001297\ Pa^4/N, \\ &\text{At } x = 0.5a, \quad w = 0.001391\ Pa^4/N. \end{aligned} \tag{5.1}$$

Along the long diagonal where $y = a - 2x$ the deflections are: (see graph No. 1)

$$\begin{aligned}
&\text{At } x = 0.1a, \quad w = 0.00012\ Pa^4/N,\\
&\text{At } x = 0.2a, \quad w = 0.00033\ Pa^4/N,\\
&\text{At } x = 0.3a, \quad w = 0.00086\ Pa^4/N,\\
&\text{At } x = 0.4a, \quad w = 0.00127\ Pa^4/N,\\
&\text{At } x = 0.5a, \quad w = 0.00139\ Pa^4/N.
\end{aligned} \tag{5.2}$$

The twisting moment is defined mathematically as:

$$M_{xy} = -N(1 - \nu)\frac{\partial^2 w}{\partial x \partial y}, \tag{5.3}$$

where w is the deflection function w_1 as formerly. Calculation of the twisting moments along the edge where $x = 0$ gives the following values: (see graph No. 2)

$$\begin{aligned}
&\text{At } y = 0, && M_{xy} = -\,0.08661\ S,\\
&\text{At } y = 0.1a, && M_{xy} = -\,0.01528\ S,\\
&\text{At } y = 0.2a, && M_{xy} = -\,0.00485\ S,\\
&\text{At } y = 0.3a, && M_{xy} = +\,0.00282\ S,\\
&\text{At } y = 0.4a, && M_{xy} = +\,0.00817\ S,\\
&\text{At } y = 0.5a, && M_{xy} = +\,0.01118\ S,\\
&\text{At } y = 0.6a, && M_{xy} = +\,0.01186\ S,\\
&\text{At } y = 0.7a, && M_{xy} = +\,0.01036\ S,\\
&\text{At } y = 0.8a, && M_{xy} = +\,0.00678\ S,\\
&\text{At } y = 0.9a, && M_{xy} = +\,0.00186\ S,\\
&\text{At } y = 1a, && M_{xy} = 0,
\end{aligned} \tag{5.4}$$

where

$$S = pa^2/(1 - \nu).$$

The twisting moment on the skew edge can be obtained from the relationship:

$$M_{ns} = (M_y - M_x)/2, \tag{5.5}$$

$$M_{ns} = -2N(1 - \nu)\,\frac{\partial^2 w}{\partial y^2},$$

where M_{ns} is taken to be the twisting moment on the skew edge, (the edge where $y = a - x$). Calculation of the twisting moment along the edge where $y = a - x$ results in the values: (see graph No. 2)

$$\begin{aligned}
&\text{At } x = 0.0a, && M_{ns} = 0,\\
&\text{At } x = 0.1a, && M_{ns} = -\,0.00995\ S,\\
&\text{At } x = 0.2a, && M_{ns} = -\,0.01640\ S,\\
&\text{At } x = 0.3a, && M_{ns} = -\,0.01800\ S,\\
&\text{At } x = 0.4a, && M_{ns} = -\,0.00484\ S,\\
&\text{At } x = 0.5a, && M_{ns} = +\,0.01448\ S,\\
&\text{At } x = 0.6a, && M_{ns} = +\,0.03900\ S,\\
&\text{At } x = 0.7a, && M_{ns} = +\,0.06888\ S,\\
&\text{At } x = 0.8a, && M_{ns} = +\,0.11192\ S,\\
&\text{At } x = 0.9a, && M_{ns} = +\,0.18776\ S,\\
&\text{At } x = 1.0a, && M_{ns} = 0,
\end{aligned} \tag{5.6}$$

where

$$S = \rho a^2(1 - \nu).$$

The solution of this problem is in terms of an infinite series which is quite rapidly convergent. The calculation of deflections and twisting moments (5.1), (5.2), (5.4), and (5.6) in most instances did not necessitate the use of more than the first three constants of (4.12) and (4.22). It was deemed sufficient in this paper to illustrate the application of the deflection function in finding the twisting moments, (5.4) and (5.6), along the edges of the plate but it should be noticed that by the correct combination of derivatives of the deflection function one may evaluate, bending moments, shears, etc., at any point in the plate.

The corner load, which is defined as the jump in the twisting moment at the corner, is zero in the 45° corners. The corner load on the 135° corners cannot be readily evaluated from the consideration of the results in graph No. 2 since in this portion of the plate the function which represents the twisting moment is not sufficiently convergent to yield a reliable result using only the first five constants.

Chapter II

Skew Plate with a Point Load at the Center of Symmetry

(6) The Plate Equation and Solution.

The 'Lagrange Plate Equation' for a point load is the homogeneous double Laplacean:

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = 0. \qquad (6.1)$$

Two functions have been constructed in such a way as to satisfy (6.1) and also contain enough arbitrary constants to satisfy the boundary conditions on the plate.

$$\begin{aligned} w_1 = \sum_{n=1,3,5\cdots} \Big\{ & A_{n_1}\left[\sinh \alpha(a - y)\sin \alpha x - \sinh \alpha x \sin \alpha y\right] + \\ & + C_{n_1}\left[(a - y)\cosh \alpha(a - y)\sin \alpha x - x\cosh \alpha x \sin \alpha y\right]\Big\}, \\ w_2 = \sum_{n=1,3,5\cdots} \Big\{ & B_{n_2}\left[\sinh \alpha(a - x)\sin \alpha y + \sinh \alpha(a + y)\sin \alpha x\right] + \\ & + D_{n_2}\left[(a - x)\cosh \alpha(a - x)\sin \alpha y + \right. \qquad (6.2) \\ & \left. + (a + y)\cosh \alpha(a + y)\sin \alpha x\right]\Big\} . \end{aligned}$$

The plate which has the same shape and dimensions as the one in Chapter I, Figure 1, will be divided into two parts by the x axis and w_1 will be valid in the region $0 \leqq y \leqq a$, while w_2 will be valid in the region $-a \leqq y \leqq 0$. It is to be noted that there is no particular integral corresponding to f_1 and f_2 of equation (1.1) for the uniformly loaded plate.

(7) Boundary Conditions and Evaluation of Constants.

The prescribed boundary conditions require that all four edges are pinned and that the deflection functions w_1 and w_2 are matched as to deflection, slope, shear, and moment on the short diagonal where $y = 0$. Mathematically these twelve conditions are:

$$M_n(w_1)\Big|_{x + y = a} = M_n(w_2)\Big|_{x + y = 0} = 0, \tag{7.1}$$

$$w_1\Big|_{x + y = a} = w_2\Big|_{x + y = 0} = 0, \tag{7.2}$$

$$w_1\Big|_{x = 0} = w_2\Big|_{x = a} = 0, \tag{7.3}$$

$$M_x\Big|_{x = 0} = M_x\Big|_{x = a} = 0, \tag{7.4}$$

$$w_1 \underset{x}{\equiv} w_2, \text{ at } y = 0, \tag{7.5}$$

$$M_y(w_1) \underset{x}{\equiv} M_y(w_2), \text{ at } y = 0, \tag{7.6}$$

$$\frac{\partial}{\partial y}\nabla^2 w_1 - \frac{\partial}{\partial y}\nabla^2 w_2 \underset{x}{\equiv} q/N, \text{ at } y = 0, \tag{7.7}$$

$$\frac{\partial w_1}{\partial y} \underset{x}{\equiv} \frac{\partial w_2}{\partial y}, \text{ at } y = 0. \tag{7.8}$$

It should be noted that (7.7) is simply the condition that the jump in shear equals the load applied at a point. The linear density of the line load is q and N is the modulus of rigidity.

Inspection of w_1 and w_2 will show that the boundary conditions, (7.1), (7.2), (7.3) and (7.4) are all fully satisfied automatically. The conditions, (7.6) and (7.7) may be simplified as was shown in (2.4)' and (2.5)'. They reduce to:

$$\frac{\partial^2 w_1}{\partial y^2} \underset{x}{\equiv} \frac{\partial^2 w_2}{\partial y^2}, \text{ at } y = 0, \tag{7.6)'}$$

$$\frac{\partial^3 w_1}{\partial y^3} - \frac{\partial^3 w_2}{\partial y^3} \underset{x}{\equiv} q/N, \text{ at } y = 0. \tag{7.7)'}$$

The condition (7.5) will yield the equation:

$$\sum_{n=1,3,5\cdots} \left\{A_{n_1} \sinh \alpha a \sin \alpha x + C_{n_1} a \cosh \alpha a \sin \alpha x\right\} \underset{x}{\equiv} \tag{7.9}$$

$$\underset{x}{\equiv} \sum_{n=1,3,5\cdots} \left\{B_{n_2} \sinh \alpha a \sin \alpha x + D_{n_2} a \cosh \alpha a \sin \alpha x\right\},$$

whence

$$A_{n_1} - B_{n_2} + a\coth n\pi(C_{n_1} - D_{n_2}) = 0. \tag{7.10}$$

The condition (7.6)' will yield the equation:

$$A_{n_1} - B_{n_2} + (a\coth n\pi + 2/\alpha)(C_{n_1} - D_{n_2}) = 0. \tag{7.11}$$

The simultaneous solution of (7.10) and (7.11) leads to the conclusion that:

$$A_{n_1} = B_{n_2}, \qquad C_{n_1} = D_{n_2}. \tag{7.12}$$

The notation may be simplified now with no ambiguity by setting:

$$A_{n_1} = A_n, \qquad C_{n_1} = C_n. \tag{7.13}$$

(8) Systems of Infinite Equations and Their Solution.

There now remain two conditions, (7.7)' and (7.8), from which we can evaluate the remaining two constants, A_n and C_n.

Consider the condition (7.7)' from which we may write the equation:

$$\sum_{n=1,3,5\ldots}\left\{2\alpha^2\left[2\cosh n\pi \sin \alpha x + \sinh \alpha x + \sinh \alpha(a - x)\right] C_n\right\} \underset{x}{\equiv} -q/N. \tag{8.1}$$

We shall consider q to be the linear density of a line load extending along the short diagonal where y = 0, (see Fig. 2) then it will be possible to

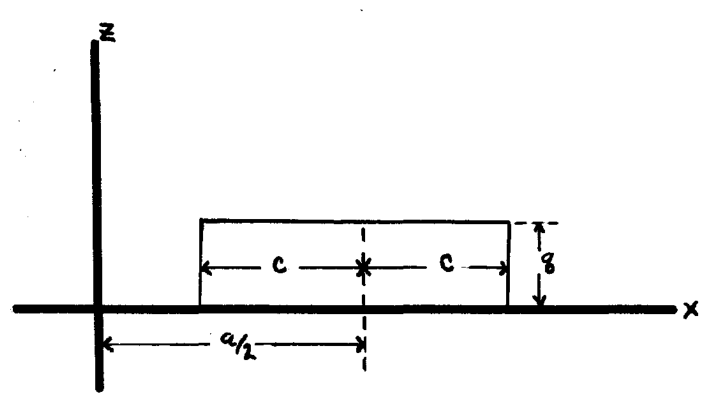


Fig. 2

expand q into a sine series in such a way that:

$$q = \frac{4q}{\pi} \sum_{n=1,3,5\ldots} \frac{(-1)^{\frac{n-1}{2}}}{n} \sin \alpha c \sin \alpha x. \tag{8.2}$$

Thus after expanding the hyperbolic functions into sine series also, one has the equation:

$$\sum_{n=1,3,5\ldots} \Big\{ \alpha^2 C_n \Big[\cosh n\pi \sin \alpha x + 2 \sinh n\pi \sum_{r=1,3,5\ldots} \frac{\beta \sin \beta x}{a(\alpha^2 + \beta^2)} \Big] + \frac{q(-1)^{\frac{n-1}{2}}}{\pi n N} \sin \alpha c \sin \alpha x \Big\} \underset{x}{\equiv} 0. \tag{8.3}$$

Let,

$$L_n = \frac{q(-1)^{\frac{n-1}{2}}}{\pi n N} \sin \alpha c, \tag{8.4}$$

and let the line loading, q, be reduced to a point load by letting c approach zero while defining $2cq = P$ where P is the desired point load. Thus:

$$\lim_{c = 0} \frac{\alpha c q (-1)^{\frac{n-1}{2}}}{\pi n N} \frac{\sin \alpha c}{\alpha c} = \frac{P(-1)^{\frac{n-1}{2}}}{2Na}, \tag{8.5}$$

or

$$L_n = \frac{P(-1)^{\frac{n-1}{2}}}{2Na} \tag{8.6}$$

It is necessary to consider the point load P at the center of the short diagonal since this symmetry allows the expansion (8.2) to be made over the odd integers and thus match the summations which we have on the deflection functions, (6.2).

In an analagous manner to that used in setting up and evaluating equations (4.5), we may solve the doubly infinite system which results from (8.3). The values obtained for the first five unknowns are:

$$N_1 = -\ 0.08054\ K,$$
$$N_3 = +\ 0.10921\ K,$$
$$N_5 = -\ 0.09705\ K,$$
$$N_7 = +\ 0.10461\ K,$$
$$N_9 = -\ 0.09851\ K,$$

where

$$N_n = n^2 C_n \sinh n\pi,$$
$$K = Pa/2N.$$

The final condition to be satisfied is the condition on the slope (7.8) and in precisely the same manner as before it leads to a doubly infinite system which involves the last set of constants as unknowns. Evaluation of the first five of these unknowns gives:

$$M_1 = +\ 0.09429\ Ka,$$
$$M_3 = -\ 0.04280\ Ka,$$
$$M_5 = +\ 0.01954\ Ka, \qquad (8.9)$$
$$M_7 = -\ 0.01633\ Ka,$$
$$M_9 = +\ 0.01085\ Ka,$$

where

$$M_n = n\sinh n\pi\, A_n,$$
$$K = Pa/2N. \tag{8.10}$$

Thus one is able to write the values for A_n and C_n:

$$
\begin{aligned}
A_1 &= +\ 8.164\ Ka \times 10^{-2}, \\
A_3 &= -\ 2.303\ Ka \times 10^{-4}, \\
A_5 &= +\ 1.178\ Ka \times 10^{-7}, \\
A_7 &= -\ 1.313\ Ka \times 10^{-10}, \\
A_9 &= +\ 1.268\ Ka \times 10^{-12},
\end{aligned}
\tag{8.11}
$$

and

$$
\begin{aligned}
C_1 &= -\ 6.974\ K \times 10^{-2}, \\
C_3 &= +\ 1.958\ K \times 10^{-5}, \\
C_5 &= -\ 1.170\ K \times 10^{-8}, \\
C_7 &= +\ 1.202\ K \times 10^{-11}, \\
C_9 &= -\ 1.278\ K \times 10^{-14}.
\end{aligned}
\tag{8.12}
$$

This completes the solution of the point load problem and it remains to calculate deflections and moments at various points in the plate.

(9) Evaluation of Deflection and Twisting Moments.

The evaluation of the deflection along the short diagonal where $y = 0$, can be rather easily carried out by using the function:

$$w = \sum_{n=1,3,5\cdots} \left[A_n \sinh n\pi + C_n a \cosh n\pi\right] \sin \alpha x. \tag{9.1}$$

The results of this calculation are: (see graph No. 1)

$$\begin{aligned}
&\text{At } x = 0, && w = 0,\\
&\text{At } x = 0.1a, && w = 0.00230\ Ka,\\
&\text{At } x = 0.2a, && w = 0.00606\ Ka,\\
&\text{At } x = 0.3a, && w = 0.01012\ Ka,\\
&\text{At } x = 0.4a, && w = 0.01395\ Ka,\\
&\text{At } x = 0.5a, && w = 0.01580\ Ka.
\end{aligned} \tag{9.2}$$

Investigation of deflections along the long diagonal yields the result that: (see graph No. 1)

$$\begin{aligned}
&\text{At } x = 0, && w = 0,\\
&\text{At } x = 0.1a, && w = 0.00006\ Ka,\\
&\text{At } x = 0.2a, && w = 0.00099\ Ka,\\
&\text{At } x = 0.3a, && w = 0.00427\ Ka,\\
&\text{At } x = 0.4a, && w = 0.01044\ Ka,\\
&\text{At } x = 0.5a, && w = 0.01580\ Ka,
\end{aligned} \tag{9.3}$$

where

$$K = Pa/2N.$$

The twisting moments along the edges, which were discussed and evaluated in Chapter I, may be evaluated here in the same manner as formerly. Along the edge where $x = 0$: (see graph No. 3)

$$
\begin{aligned}
&\text{At } y = 0, && M_{xy} = -\,0.66256\ R,\\
&\text{At } y = 0.1a, && M_{xy} = -\,0.04368\ R,\\
&\text{At } y = 0.2a, && M_{xy} = +\,0.03342\ R,\\
&\text{At } y = 0.3a, && M_{xy} = +\,0.06590\ R,\\
&\text{At } y = 0.4a, && M_{xy} = +\,0.07134\ R,\\
&\text{At } y = 0.5a, && M_{xy} = +\,0.06263\ R,\\
&\text{At } y = 0.6a, && M_{xy} = +\,0.04620\ R,\\
&\text{At } y = 0.7a, && M_{xy} = +\,0.02886\ R,\\
&\text{At } y = 0.8a, && M_{xy} = +\,0.01337\ R,\\
&\text{At } y = 0.9a, && M_{xy} = +\,0.00345\ R,\\
&\text{At } y = a, && M_{xy} = 0,
\end{aligned}
\tag{9.4}
$$

where $R = NK(1-\nu)/a$. And along the edge where $x + y = a$: (see graph No. 3)

$$
\begin{aligned}
&\text{At } x = 0, && M_{ns} = 0,\\
&\text{At } x = 0.1a, && M_{ns} = -\,0.01368\ R,\\
&\text{At } x = 0.2a, && M_{ns} = -\,0.05168\ R,\\
&\text{At } x = 0.3a, && M_{ns} = -\,0.10656\ R,\\
&\text{At } x = 0.4a, && M_{ns} = -\,0.16740\ R,\\
&\text{At } x = 0.5a, && M_{ns} = -\,0.21740\ R,\\
&\text{At } x = 0.6a, && M_{ns} = -\,0.16112\ R,\\
&\text{At } x = 0.7a, && M_{ns} = -\,0.09588\ R,\\
&\text{At } x = 0.8a, && M_{ns} = +\,0.11620\ R,\\
&\text{At } x = 0.9a, && M_{ns} = +\,0.28468\ R,\\
&\text{At } x = a, && M_{ns} = 0.
\end{aligned}
\tag{9.5}
$$

(10) Experimental Verification[12]

A plate of the particular shape in Fig. 1, was cut from a sheet of boiler plate of thickness 0.25 inches. The dimension, a, of Fig. 1 was 23.75 inches. A load was applied at the center of symmetry thru a circular bearing surface of radius three-eights of an inch. The load in magnitude ranged between zero and 2000 pounds, approximately, while the center deflection ranged from zero to about 0.25 inches. The deflection of points on the diagonals corresponding to the points plotted in graph No. 1 were found in all cases to check within 5 per cent of the calculated values.

Corner loads were measured by the use of springs and it was found that altho all four corners kicked up unless they were held down, the load required to hold down the 45° corners was about one-half of one per cent of the center load while the load required on the 135° corners did not exceed five per cent of the center load.

It was interesting to note that when the corners were forced down into contact with the supports the edges of the plate came into contact with the supports everywhere. Most of the load seemed to be carried by the portion of the supports nearest the load. Whether the 45° corner was forced down or allowed to kick up did not seem to influence the deflections of the plate near the center.

Chapter III

Sector-Shaped Plates

(11) Plate Equation and Interior Conditions.

The problem which involves finding the stresses and strains in a sector shaped plate loaded with a uniform load has been discussed by Galerkin.[13] This chapter deals with the problem of a sector shaped plate such as the one in Fig. 3. The angle, γ, of the sector in all cases may vary from 0 to π. A vertical point load is located at the point (r_1, θ_1) and the radial edges are simply supported. The problem is solved for four distinct boundary conditions on the circular edge namely: pinned edge, fixed edge, free edge, and Navier edge.

Consider the problem of a finite sector-shaped plate, as illustrated in Fig. 3. Construct two deflection functions, w_1 and w_2, such that:

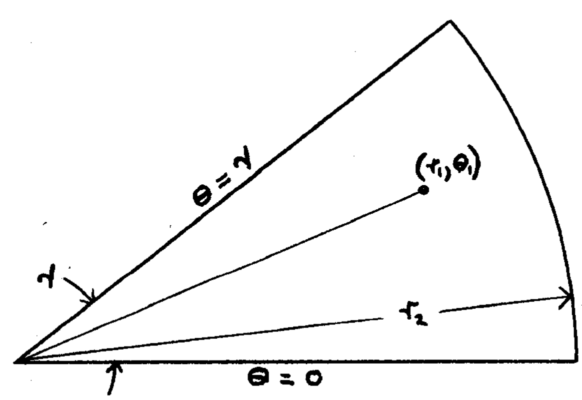


Fig. 3

$$w_1 \text{ is valid for } r \leq r_1,$$
$$w_2 \text{ is valid for } r_1 \leq r \leq r_2, \tag{11.1}$$

$$w_1 = \sum_{n=1}^{\infty} \left\{ A_n r^{\alpha} + C_n r^{2+\alpha} \right\} \sin \alpha\theta,$$
$$w_2 = \sum_{n=1}^{\infty} \left\{ E_n r^{\alpha} + F_n r^{-\alpha} + G_n r^{2+\alpha} + H_n r^{2-\alpha} \right\} \sin \alpha\theta, \tag{11.2}$$

where

$$\alpha = n\pi/\gamma.$$

It is easy to show that these deflection functions satisfy the plate equation for a point load:

$$\nabla^4 w = 0. \tag{11.3}$$

They also automatically satisfy the conditions which make the radial edges pinned, namely:

$$w_1 = w_2 = 0, \quad \text{at } \theta = \gamma,$$
$$w_1 = w_2 = 0, \quad \text{at } \theta = 0, \tag{11.4}$$
$$\frac{\partial^2 w_1}{\partial \theta^2} = \frac{\partial^2 w_2}{\partial \theta^2} = 0, \quad \text{at } \theta = \gamma,$$
$$\frac{\partial^2 w_1}{\partial \theta^2} = \frac{\partial^2 w_2}{\partial \theta^2} = 0, \quad \text{at } \theta = 0.$$

It remains to evaluate the six arbitrary constants in (11.2) by first matching the deflection functions, (11.2), at the arc where $r = r_1$ and secondly meeting whatever requirements are to be placed on the outer circular edge where $r = r_2$.

The deflection functions, (11.2), must be matched at $r = r_1$ in the following way:

$$w_1 \underset{\theta}{\equiv} w_2, \quad \text{at } r = r_1, \tag{11.5}$$

$$\frac{\partial w_1}{\partial r} \underset{\theta}{\equiv} \frac{\partial w_2}{\partial r}, \quad \text{at } r = r_1, \tag{11.6}$$

$$M_r(w_1) \underset{\theta}{\equiv} M_r(w_2), \quad \text{at } r = r_1, \tag{11.7}$$

$$V_r(w_2) - V_r(w_1) = -q, \quad \text{at } r = r_1, \tag{11.8}$$

but it is easily shown since:

$$M_r = -N\left\{\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right\},$$

and

$$V_r = -N(1-\nu)\left\{\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} + \frac{\partial^3 w}{\partial r \partial \theta^2}\right\},$$

that the conditions (11.7) and (11.8) may be replaced by the equivalent conditions:

$$\frac{\partial^2 w_1}{\partial r^2} \underset{\theta}{\equiv} \frac{\partial^2 w_2}{\partial r^2}, \quad \text{at } r = r_1, \tag{11.7'}$$

$$\frac{\partial^3 w_2}{\partial r^3} - \frac{\partial^3 w_1}{\partial r^3} \underset{\theta}{\equiv} q/N, \quad \text{at } r = r_1. \tag{11.8'}$$

The deflection condition, (11.5), leads to the equation:

$$\sum_{n=1}^{\infty}\left\{(A_n - E_n)r_1^{\alpha} + (C_n - G_n)r_1^{2+\alpha} - F_n r_1^{-\alpha} - H_n r_1^{2-\alpha}\right\}\sin\alpha\theta \underset{\theta}{\equiv} \underset{\theta}{\equiv} 0.$$

The only way this identity can be true for all values of θ is that

$$(A_n - E_n)r_1^{\alpha} + (C_n - G_n)r_1^{2+\alpha} - F_n r_1^{-\alpha} - H_n r_1^{2-\alpha} = 0. \qquad (11.9)$$

$$n = 1,2,3,4,\ldots.$$

The slope condition (11.6) leads to the equation:

$$\sum_{n=1}^{\infty}\left\{\alpha(A_n - E_n)r_1^{\alpha-1} + (2+\alpha)(C_n - G_n)r_1^{1+\alpha} + \alpha F_n r_1^{-\alpha-1} - - (2-\alpha)\,H_n r_1^{1-\alpha}\right\}\sin\alpha\theta \underset{\theta}{\equiv} 0.$$

This identity holds for all values of θ if

$$\alpha(A_n - E_n)r_1^{\alpha-1} + (2+\alpha)(C_n - G_n)r_1^{1+\alpha} + \alpha F_n r_1^{-\alpha-1} - - (2-\alpha)H_n r_1^{1-\alpha} = 0. \qquad (11.10)$$

$$n = 1,2,3,4,\ldots.$$

The moment condition (11.7)' leads to the equation:

$$\sum_{n=1}^{\infty}\Big\{\alpha(\alpha - 1)(A_n - E_n)r_1^{\alpha-2} + (2 + \alpha)(1 + \alpha)(C_n - G_n)r_1^{\alpha} -$$

$$- \alpha(\alpha + 1)F_n r_1^{-\alpha-2} - (2 - \alpha)(1 - \alpha)H_n r_1^{-\alpha}\Big\}\sin \alpha\theta \underset{\theta}{\equiv} 0.$$

whence to satisfy the identity in θ:

$$\alpha(\alpha - 1)(A_n - E_n)r_1^{\alpha-2} + (2 + \alpha)(1 + \alpha)(C_n - G_n)r_1^{\alpha} -$$

$$- \alpha(\alpha + 1)F_n r_1^{-\alpha-2} - (2 - \alpha)(1 - \alpha)H_n r_1^{-\alpha} = 0. \quad (11.11)$$

$$n = 1,2,3,4,\ldots.$$

The shear condition (11.8)' leads to the equation:

$$\sum_{n=1}^{\infty}\Big\{\alpha(\alpha - 1)(\alpha - 2)(A_n - E_n)r_1^{\alpha-3} + \alpha(1 + \alpha)(2 + \alpha)(C_n -$$

$$- G_n)r_1^{\alpha-1} + \alpha(\alpha + 1)(\alpha + 2)F_n r_1^{-\alpha-3} + (2 - \alpha)(1 -$$

$$- \alpha)\alpha H_n r_1^{-\alpha-1}\Big\}\sin \alpha\theta \underset{\theta}{\equiv} -q/N. \quad (11.12)$$

It becomes necessary now to expand the right hand side of the equation (11.12) into a sine series. If one employs the unit expansion:

$$q = \frac{4q}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin \alpha\theta_1 \sin \alpha\theta \sin \alpha\varphi, \quad (11.13)$$

where q is the intensity of load per unit length of arc as illustrated in Fig. 4, then a limiting process may be used to shrink this line loading down to a point load P at (r_1, θ_1).

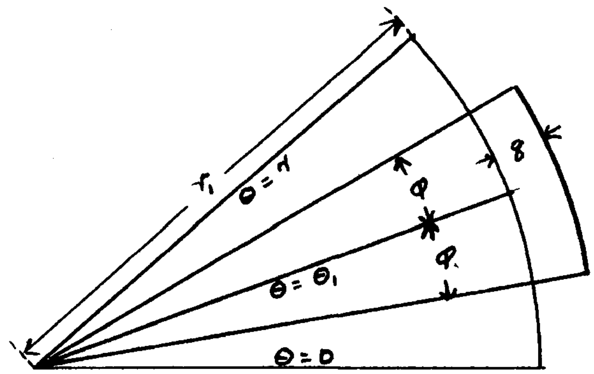


Fig. 4

Define:

$$P = \lim_{\varphi \to 0} 2r_1 \varphi q = \text{point load.}$$

Now consider:

$$q \sin \alpha\varphi = q\alpha\varphi \frac{\sin \alpha\varphi}{\alpha\varphi} = \frac{\alpha}{2r_1}\left(2r_1\varphi q \frac{\sin \alpha\varphi}{\alpha\varphi}\right),$$

then

$$\lim_{\varphi \to 0} \frac{\alpha}{2r_1}\left(2r_1\varphi q \frac{\sin \alpha\varphi}{\alpha\varphi}\right) = \frac{\alpha P}{2r_1} . \qquad (11.14)$$

Equation (11.12) can now be written:

$$\sum_{n=1}^{\infty}\Big\{\alpha(\alpha - 1)(\alpha - 2)(A_n - E_n)r_1^{\alpha-3} + \alpha(1 + \alpha)(2 + \alpha)(C_n - G_n)r_1^{\alpha-1} + \alpha(\alpha + 1)(\alpha + 2)F_n r_1^{-\alpha-3} + (2 - \alpha)(1 - \alpha)\alpha H_n r_1^{-\alpha-1}\Big\}\sin \alpha\theta \underset{\theta}{=} \sum_{n=1}^{\infty} - \frac{2P\alpha \sin \alpha\theta_1}{\pi n N r_1} \sin \alpha\theta .$$

In order to be an identity in θ we have:

$$\alpha(\alpha - 1)(\alpha - 2)(A_n - E_n)r_1^{\alpha-3} + \alpha(1 + \alpha)(2 + \alpha)(C_n - G_n)r_1^{\alpha-1} + \alpha(\alpha + 1)(\alpha + 2)F_n r_1^{-\alpha-3} + (2 - \alpha)(1 - \alpha)\alpha H_n r_1^{-\alpha-1} = -\frac{2P\alpha \sin \alpha\theta_1}{n\pi N r_1}. \tag{11.15}$$

$$n = 1,2,3,4,\ldots.$$

The four equations (11.9), (11.10), (11.11), and (11.15) may now be solved to evaluate $(A_n - E_n)$, $(C_n - G_n)$, F_n, and H_n.

$$\begin{aligned} A_n - E_n &= -\frac{Pr_1^{2-\alpha}\sin \alpha\theta_1}{4\pi nN(1 - \alpha)}, \\ C_n - G_n &= -\frac{Pr_1^{-\alpha}\sin \alpha\theta_1}{4\pi nN(1 + \alpha)}, \\ F_n &= -\frac{Pr_1^{2+\alpha}\sin \alpha\theta_1}{4\pi nN(1 + \alpha)}, \\ H_n &= -\frac{Pr_1^{\alpha}\sin \alpha\theta_1}{4\pi nN(1 - \alpha)}. \end{aligned} \tag{11.16}$$

(12) Plate with Pinned, Fixed or Free Circular Edge.

In order to satisfy the requirements of a pinned edge at $r = r_2$ it is necessary that:

$$M_r(w_2) = w_2 = 0, \quad \text{at } r = r_2. \tag{12.1}$$

The deflection condition requires that:

$$\sum_{n=1}^{\infty}\left\{E_n r_2^{\alpha} + F_n r_2^{-\alpha} + G_n r_2^{2+\alpha} + H_n r_2^{2-\alpha}\right\}\sin\alpha\theta \underset{\theta}{\equiv} 0,$$

or

$$E_n r_2^{\alpha} + F_n r_2^{-\alpha} + G_n r_2^{2+\alpha} + H_n r_2^{2-\alpha} = 0. \qquad (12.2)$$

$$n = 1,2,3,4,\ldots.$$

The moment condition requires that:

$$\sum_{n=1}^{\infty}\left\{\alpha(\alpha - 1)(1 - \nu)E_n r_2^{\alpha-2} + \alpha(\alpha + 1)(1 - \nu)F_n r_2^{-\alpha-2} + (1 + \alpha)\left[2(1 + \nu) + \alpha(1 - \nu)\right]G_n r_2^{\alpha} + (1 - \alpha)\left[2(1 + \nu) - \alpha(1 - \nu)\right]H_n r_2^{-\alpha}\right\}\sin\alpha\theta \underset{\theta}{\equiv} 0,$$

or

$$\alpha(\alpha - 1)(1 - \nu)E_n r_2^{\alpha-2} + \alpha(\alpha + 1)(1 - \nu)F_n r_2^{-\alpha-2} + (1 + \alpha)\left[2(1 + \nu) + \alpha(1 - \nu)\right]G_n r_2^{\alpha} + (1 - \alpha)\left[2(1 + \nu) - \alpha(1 - \nu)\right]H_n r_2^{-\alpha} = 0. \qquad (12.3)$$

$$n = 1,2,3,4,\ldots.$$

Since the value of F_n and H_n is known from (11.16), the solution of (12.2) and (12.3) gives:

$$E_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi mNr_2^{2\alpha}(1 + \nu + 2\alpha)}\left\{(1 + \nu)r_1^2 + \frac{\alpha(3 + \nu)}{1 - \alpha}r_2^2\right\},$$

$$G_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi mNr_2^{2\alpha+2}(1 + \nu + 2\alpha)}\left\{\frac{\alpha(1 - \nu)}{1 + \alpha}r_1^2 + (1 + \nu)r_2^2\right\}. \qquad (12.4)$$

The constants for the case of a pinned circular arc are therefore:

$$A_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi mNr_2^{2\alpha}}\left\{\frac{1+\nu}{1+\nu+2\alpha}r_1^2 + \frac{\alpha(3+\nu)}{(1-\alpha)(1+\nu+2\alpha)}r_2^2 - \frac{1}{1-\alpha}r_2^{2\alpha}r_1^{2-2\alpha}\right\},$$

$$C_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi mNr_2^{2\alpha+2}}\left\{\frac{\alpha(1-\nu)}{(1+\alpha)(1+\nu+2\alpha)}r_1^2 + \frac{1+\nu}{1+\nu+2\alpha}r_2^2 - \frac{1}{1+\alpha}r_2^{2\alpha+2}r_1^{-2\alpha}\right\},$$

$$E_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi mNr_2^{2\alpha}}\left\{\frac{1+\nu}{1+\nu+2\alpha}r_1^2 + \frac{\alpha(3+\nu)}{(1-\alpha)(1+\nu+2\alpha)}r_2^2\right\},$$

$$F_n = -\frac{Pr_1^{\alpha+2}\sin\alpha\theta_1}{4\pi mN(1+\alpha)}, \tag{12.5}$$

$$G_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi mNr_2^{2\alpha+2}}\left\{\frac{\alpha(1-\nu)}{(1+\alpha)(1+\nu+2\alpha)}r_1^2 + \frac{1+\nu}{1+\nu+2\alpha}r_2^2\right\},$$

$$H_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi mN(\alpha-1)}.$$

For the case of a <u>fixed circular edge</u> the requirements to be met are:

$$\frac{\partial w_2}{\partial r} = w_2 = 0, \quad \text{at } r = r_2 . \tag{12.6}$$

The deflection requirement is satisfied by equation (12.2) while a zero slope at $r = r_2$ is assured if:

$$\sum_{n=1}^{\infty}\left\{\alpha E_n r_2^{\alpha-1} - \alpha F_n r_2^{-\alpha-1} + (2+\alpha)G_n r_2^{\alpha+1} + (2 - \right.$$

$$\left. - \alpha)H_n r_2^{1-\alpha}\right\}\sin\alpha\theta \underset{\theta}{\equiv} 0.$$

or

$$\alpha E_n r_2^{\alpha-1} - \alpha F_n r_2^{-\alpha-1} + (2+\alpha)G_n r_2^{\alpha+1} + (2 -$$

$$- \alpha)H_n r_2^{1-\alpha} = 0. \tag{12.7}$$

Equations (12.2) and (12.7) may be solved for E_n and G_n after insertion of the values for F_n and H_n from (11.16). When E_n and G_n have been evaluated one may find A_n and C_n from equations (11.16). Thus the constants for the fixed circular edge condition become:

$$A_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi mNr_2^{2\alpha}}\left\{r_1^2 + \frac{\alpha}{1-\alpha}r_2^2 - \frac{1}{1-\alpha}r_2^{2\alpha}r_1^{2-2\alpha}\right\},$$

$$C_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi mNr_2^{2\alpha+2}}\left\{-\frac{\alpha}{1+\alpha}r_1^2 + r_2^2 - \frac{1}{1+\alpha}r_2^{2\alpha+2}r_1^{-2\alpha}\right\},$$

$$E_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi mNr_2^{2\alpha}}\left\{r_1^2 + \frac{\alpha}{1-\alpha}r_2^2\right\}, \tag{12.8}$$

$$F_n = -\frac{Pr_1^{\alpha+2}\sin\alpha\theta_1}{4\pi mN(1+\alpha)},$$

$$G_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi mNr_2^{2\alpha+2}}\left\{-\frac{\alpha}{1+\alpha}r_1^2 + r_2^2\right\},$$

$$H_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi nN(\alpha - 1)} .$$

A <u>free edge</u> is an edge at which there is no support of any kind. The plate may deflect or twist with no exterior restraint along such an edge. The mathematical conditions are that the reaction and the normal bending moment vanish at each point along the edge.

$$R_n(w_2) = M_n(w_2) = 0, \quad \text{at } r = r_2, \tag{12.9}$$

where:

$$R_n(w) = V_r + \frac{1}{r}\frac{\partial}{\partial\theta} M_{r\theta}$$

$$= -N\left\{\frac{\partial}{\partial r}\nabla^2 w + \frac{(1-\nu)}{r}\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\right]\right\} . \tag{12.10}$$

The bending moment condition yields equation (12.3) and the reaction condition requires:

$$\sum_{n=1}^{\infty}\Big\{-\alpha^2(\alpha - 1)(1 - \nu)E_n r_2^{\alpha-3} + \alpha^2(\alpha + 1)(1 - \nu)F_n r_2^{-\alpha-3} +$$

$$+ \alpha(1 + \alpha)\left[4 - (1 - \nu)\alpha\right]G_n r_2^{\alpha-1} - \alpha(1 - \alpha)\left[4 + (1 -\right.$$

$$\left.-\nu)\alpha\right]H_n r_2^{-\alpha-1}\Big\}\sin\alpha\theta \underset{\theta}{\equiv} 0,$$

or

$$\alpha^2(\alpha - 1)(1 - \nu)E_n r_2^{\alpha-2} - \alpha^2(\alpha + 1)(1 - \nu)F_n r_2^{-\alpha-2} -$$

$$- \alpha(1 + \alpha)\left[4 - (1 - \nu)\alpha\right]G_n r_2^{\alpha} + \alpha(1 - \alpha)\left[4 + (1 -\right.$$

$$\left.-\nu)\alpha\right]H_n r_2^{-\alpha} = 0. \tag{12.11}$$

Thus, as in the pinned and fixed edge cases, we may solve for the constants which result from equations (11.16), (12.3), and (12.11). Accordingly the constants for the free edge case are:

$$A_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi nNr_2^{2\alpha}}\left\{\frac{8(1+\nu)+\alpha^2(1-\nu)^2}{\alpha(\alpha-1)(1-\nu)(3+\nu)}\,r_2^2 - \right.$$

$$\left. - \frac{1-\nu}{3+\nu}r_1^2 - \frac{1}{1-\alpha}\,r_2^{2\alpha}r_1^{2-2\alpha}\right\},$$

$$C_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi nNr_2^{2\alpha+2}}\left\{\frac{\alpha(1-\nu)}{(1+\alpha)(3+\nu)}\,r_1^2 - \frac{1-\nu}{3+\nu}\,r_2^2 - \right.$$

$$\left. - \frac{1}{1+\alpha}\,r_2^{2\alpha+2}r_1^{-2\alpha}\right\},$$

$$E_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi nNr_2^{2\alpha}}\left\{-\frac{(1-\nu)}{3+\nu}\,r_1^2 + \frac{8(1+\nu)+\alpha^2(1-\nu)^2}{\alpha(\alpha-1)(1-\nu)(3+\nu)}r_2^2\right\},$$

$$\tag{12.12}$$

$$F_n = -\frac{Pr_1^{2+\alpha}\sin\alpha\theta_1}{4\pi nN(1+\alpha)},$$

$$G_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi nNr_2^{2\alpha+2}}\left\{\frac{\alpha(1-\nu)}{(1+\alpha)(3+\nu)}\,r_1^2 - \frac{1-\nu}{3+\nu}\,r_2^2\right\},$$

$$H_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi nN(\alpha-1)}.$$

(13) <u>Navier Circular Edge</u>.

Since the Laplacean is proportional to the moment sum, M_e:

$$M_e = M_r + M_\theta = -N(1 + \nu)\nabla^2 w, \qquad (13.1)$$

the Navier edge condition may be precisely described as an edge which is supported in such a manner that the edge suffers no deflection and at which the moment sum, M_e, vanishes. Along a rectilinear edge the Navier edge condition is equivalent to the pinned edge condition. (see page 11) Mathematically the Navier condition may be written:

$$w_2 = \nabla^2 w_2 = 0, \quad \text{at } r = r_2. \qquad (13.2)$$

The deflection requirement leads to equation (12.2), and the vanishing of the Laplacean, or moment sum yields:

$$\sum_{n=1}^{\infty}\left\{(1 + \alpha)G_n r_2^{\alpha} + (1 - \alpha)H_n r_2^{-\alpha}\right\}\sin \alpha\theta \underset{\theta}{\equiv} 0,$$

or

$$(1 + \alpha)G_n r_2^{\alpha} + (1 - \alpha)H_n r_2^{-\alpha} = 0. \qquad (13.3)$$

The solution of equations (12.2) and (13.3) with the aid of equations (11.16) gives the following values for the constants:

$$A_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi nNr_2^{2\alpha}}\left\{\frac{1}{1+\alpha}r_1^2 + \frac{2\alpha}{1-\alpha^2}r_2^2 - \right.$$

$$\left. - \frac{1}{1-\alpha}r_2^{2\alpha}r_1^{2-2\alpha}\right\},$$

$$C_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi nNr_2^{2\alpha+2}}\left\{\frac{1}{1+\alpha}r_2^2 - \frac{1}{1+\alpha}r_2^{2\alpha+2}r_1^{-2\alpha}\right\},$$

$$E_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi nNr_2^{2\alpha}}\left\{\frac{1}{1+\alpha}r_1^2 + \frac{2\alpha}{1-\alpha^2}r_2^2\right\},$$

$$F_n = -\frac{Pr_1^{2+\alpha}\sin\alpha\theta_1}{4\pi nN(1+\alpha)}, \qquad (13.4)$$

$$G_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi nN(1+\alpha)r_2^{2\alpha}},$$

$$H_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi nN(\alpha-1)}.$$

The moment sum (13.1) for this case may be written in closed form. Consider the Laplacean of the deflection functions (11.2) after the constants, (13.4), have been inserted:

$$\nabla^2 w_1 = \frac{P}{\pi N}\sum_{n=1}^{\infty}\frac{1}{n}(u^{\alpha} - R^{\alpha})\sin\alpha\theta_1\sin\alpha\theta,$$
$$\nabla^2 w_2 = \frac{P}{\pi N}\sum_{n=1}^{\infty}\frac{1}{n}(u^{\alpha} - R^{-\alpha})\sin\alpha\theta_1\sin\alpha\theta, \qquad (13.5)$$

where

$$u = rr_1/r_2^2,$$
$$R = r/r_1. \tag{13.6}$$

Since one may show[14] that

$$\sum_{m=1}^{\infty} \frac{a^m}{m} \sin mx \sin my = \frac{1}{4} \log \frac{a + a^{-1} - 2\cos(x + y)}{a + a^{-1} - 2\cos(x - y)}, \tag{13.7}$$

in the region where $a \leqq 1$ it follows directly that both equations (13.5) will reduce to:

$$\nabla^2 w = \frac{P}{4\pi N} \log \frac{u^k + u^{-k} - 2\cos(\theta_1 + \theta)}{u^k + u^{-k} - 2\cos(\theta_1 - \theta)} - \log \frac{R^k + R^{-k} - 2\cos(\theta_1 + \theta)}{R^k + R^{-k} - 2\cos(\theta_1 - \theta)} . \tag{13.8}$$

It follows readily from (13.1) that:

$$M_o = \frac{P(1 + \nu)}{4\pi} \log\left\{\frac{u^k + u^{-k} - 2\cos(\theta_1 - \theta)}{u^k + u^{-k} - 2\cos(\theta_1 + \theta)} \cdot \frac{R^k + R^{-k} - 2\cos(\theta_1 + \theta)}{R^k + R^{-k} - 2\cos(\theta_1 - \theta)}\right\}, \tag{13.9}$$

where $k = \pi/\gamma$ and the function (13.9) is valid in the entire region $0 \leqq r \leqq r_2$.

(14) Solution in Closed Form[15] for Infinite Sector.

If r_2 is allowed to increase to infinity all of the solutions for the special cases of a finite sector plate will immediately reduce from (12.5), (12.8), (12.12) or (13.4) to:

$$A_n = \frac{Pr_1^{2-\alpha}\sin\alpha\theta_1}{4\pi n N(\alpha - 1)},$$

$$C_n = -\frac{Pr_1^{-\alpha}\sin\alpha\theta_1}{4\pi n N(\alpha + 1)}, \tag{14.1}$$

$$F_n = -\frac{Pr_1^{2+\alpha}\sin\alpha\beta_1}{4\pi n N(\alpha + 1)},$$

$$H_n = \frac{Pr_1^{\alpha}\sin\alpha\theta_1}{4\pi n N(\alpha - 1)},$$

$$E_n = G_n = 0 .$$

The deflection functions (11.2) for an infinite sector-shaped plate then become:

$$w_1 = \frac{Pr_1^2}{4\pi N}\sum_{n=1}^{\infty} 1/n\left\{\frac{R^{\alpha}}{\alpha - 1} - \frac{R^{2+\alpha}}{\alpha + 1}\right\}\sin\alpha\theta_1\sin\alpha\theta ,$$

$$w_2 = \frac{Pr_1^2}{4\pi N}\sum_{n=1}^{\infty} 1/n\left\{-\frac{R^{-\alpha}}{\alpha + 1} + \frac{R^{2-\alpha}}{\alpha - 1}\right\}\sin\alpha\theta_1\sin\alpha\theta, \tag{14.2}$$

where $R = r/r_1$.

Consider the Laplacean of these functions, (14.2), which may be written:

$$\nabla^2 w_1 = -\frac{P}{\pi N}\sum_{n=1}^{\infty}\frac{R^{\alpha}}{n}\sin\alpha\theta_1\sin\alpha\theta,$$
$$\nabla^2 w_2 = -\frac{P}{\pi N}\sum_{n=1}^{\infty}\frac{R^{-\alpha}}{n}\sin\alpha\theta_1\sin\alpha\theta. \qquad (14.3)$$

The relation (13.7) allows one to write either of the quantities (14.3) in the closed form:

$$\nabla^2 w = \frac{P}{4\pi N}\log\frac{R^k + R^{-k} - 2\cos k(\theta_1 - \theta)}{R^k + R^{-k} - 2\cos k(\theta_1 + \theta)}, \qquad (14.4)$$

where $k = \pi/\gamma$. It follows then that the moment sum may be written as:

$$M_o = \frac{P(1+\nu)}{4\pi}\log\frac{R^k + R^{-k} - 2\cos k(\theta_1 + \theta)}{R^k + R^{-k} - 2\cos k(\theta_1 - \theta)}. \qquad (14.5)$$

One may readily verify that the limit of the moment sum, (13.9), as r_2 approaches infinity, is the moment sum, (14.5), for the infinite plate.

The bending moments and the twisting moment in the infinite plate may also be written in closed form as follows:

$$M_r = \frac{M_o}{2} + \frac{1-\nu}{4(1+\nu)}\left(R - \frac{1}{R}\right)\frac{\partial M_o}{\partial R}, \qquad (14.6)$$

$$M_\theta = \frac{M_o}{2} - \frac{1 - \nu}{4(1 + \nu)}\left(R - \frac{1}{R}\right)\frac{\partial M_o}{\partial R}\ , \tag{14.7}$$

$$M_{r\theta} = \frac{1 - \nu}{4(1 + \nu)}\left(1 - \frac{1}{R^2}\right)\frac{\partial M_o}{\partial \theta}\ . \tag{14.8}$$

All of these functions (14.5), (14.6), (14.7), and (14.8), are valid for all parts of the plate.

(15) Application of Complex Conformal Transformation to the Sector-Shaped Plate.

It is an interesting verification of the expressions (13.9) and (14.5) to be able to derive them by a completely different method than the one which has been used thus far. One may construct a mapping function which will map the finite sector of Fig. 3 in a one to one manner into the interior of a unit circle with the load point, (r_1, θ_1) going into the center of the unit circle. This transformation may be written:

$$\zeta = \frac{\left(\dfrac{1 + (z_1/r_2)^k}{1 - (z_1/r_2)^k}\right)^2 - \left(\dfrac{1 + (z/r_2)^k}{1 - (z/r_2)^k}\right)^2}{\left(\dfrac{1 + (z/r_2)^k}{1 - (z/r_2)^k}\right)^2 - \left(\dfrac{1 + (\bar{z}_1/r_2)^k}{1 - (\bar{z}_1/r_2)^k}\right)^2}\ , \tag{15.1}$$

where

$$
\begin{aligned}
\zeta &= \rho e^{i\lambda}, \\
z_1 &= r_1 e^{i\theta_1}, \\
\bar{z}_1 &= r_1 e^{-i\theta_1}, \\
z &= re^{i\theta}, \\
k &= \pi/\gamma.
\end{aligned}
\tag{15.2}
$$

This transformation (15.1) may be factored into three phases. First the transformation which opens the original sector into a semicircle:

$$z' = (z/r_2)^k, \tag{15.3}$$

then the transformation which takes this semicircle of unit radius into an infinite half-plane:

$$z'' = \left(\frac{1 + z'}{1 - z'}\right)^2, \tag{15.4}$$

and finally the transformation which puts the infinite half-plane on the interior of a unit circle whose center corresponds to (r_1, θ_1):

$$\zeta = \frac{z_1'' - z''}{z'' - \bar{z}_1''}. \tag{15.5}$$

In order to find the absolute value of

(15.1) one may proceed as follows:

$$\zeta = \frac{\dfrac{1+(z_1/r_2)^k}{1-(z_1/r_2)^k} - \dfrac{1+(z/r_2)^k}{1-(z/r_2)^k}}{\dfrac{1+(z/r_2)^k}{1-(z/r_2)^k} - \dfrac{1+(\bar{z}_1/r_2)^k}{1-(\bar{z}_1/r_2)^k}} \cdot \frac{\dfrac{1+(z_1/r_2)^k}{1-(z_1/r_2)^k} + \dfrac{1+(z/r_2)^k}{1-(z/r_2)^k}}{\dfrac{1+(z/r_2)^k}{1-(z/r_2)^k} + \dfrac{1+(\bar{z}/r_2)^k}{1-(\bar{z}_1/r_2)^k}}, \tag{15.6}$$

and

$$\frac{1+(z_1/r_2)^k}{1-(z_1/r_2)^k} - \frac{1+(z/r_2)^k}{1-(z/r_2)^k} = \tag{15.7}$$

$$= \frac{s_1e^{ik\theta_1} - se^{ik\theta}}{1 - se^{ik\theta} - s_1e^{ik\theta_1} + ss_1e^{ik(\theta+\theta_1)}},$$

where

$$s = (r/r_2)^k,$$
$$s_1 = (r_1/r_2)^k. \tag{15.8}$$

If the identity

$$e^{ik\theta} = \cos k\theta + i\sin k\theta \tag{15.9}$$

is used in writing out (15.7), and if the absolute values of the numerator and denominator are calculated independently one has:

$$\frac{\left[s_1^2 + s^2 - 2ss_1\cos k(\theta_1 - \theta)\right]^{1/2}}{F(s,s_1,k,\theta,\theta_1)} . \tag{15.10}$$

Similarly

$$\frac{1 + (z/r_2)^k}{1 - (z/r_2)^k} - \frac{1 + (\bar{z}_1/r_2)^k}{1 - (\bar{z}_1/r_2)^k} = \tag{15.11}$$

$$= \frac{\left[s_1^2 + s^2 - 2ss_1\cos k(\theta_1 + \theta)\right]^{1/2}}{F(s,s_1,k,\theta,\theta_1)} ,$$

whence the absolute value of the first factor of (15.6) becomes:

$$\left\{\frac{s_1^2 + s^2 - 2ss_1\cos k(\theta_1 - \theta)}{s_1^2 + s^2 - 2ss_1\cos k(\theta_1 + \theta)}\right\}^{1/2} . \tag{15.12}$$

The absolute value of the second factor of (15.6) by a similar procedure reduces to:

$$\left\{\frac{1 + s^2s_1^2 - 2ss_1\cos k(\theta_1 + \theta)}{1 + s^2s_1^2 - 2ss_1\cos k(\theta_1 - \theta)}\right\}^{1/2} . \tag{15.13}$$

In terms of the notation (13.6) it is evident that

$$\rho = \left\{ \frac{R^k + R^{-k} - 2\cos k(\theta_1 - \theta)}{R^k + R^{-k} - 2\cos k(\theta_1 + \theta)} \cdot \frac{u^k + u^{-k} - 2\cos k(\theta_1 + \theta)}{u^k + u^{-k} - 2\cos k(\theta_1 - \theta)} \right\}^{1/2}. \tag{15.14}$$

One may now construct a Green's Function[16] from (15.14) by writing $\log \rho$. This function, $\log \rho$, has the properties:

(1). Since the boundary of the sector transformed into the boundary of the circle and since $\rho = 1$ on the boundary of the circle, this function vanishes at all points on the periphery.

(2). Since the load point of the sector becomes the origin of the unit circle this Green's Function becomes infinite like $\log \rho$, as ρ approaches zero, at the load point.

(3). Since ρ is the absolute value of an analytic function it is clear that $\log \rho$ is a harmonic function.

These are precisely the conditions which must be fulfilled by the moment sum function, M_o, on the sector-shaped slab which is pinned on its radial edges and has a Navier condition at the circular arc. Therefore consider the plate equation (11.3) and write it:

$$\nabla^2 \nabla^2 w = 0. \tag{15.15}$$

From (13.1) we know:

$$\nabla^2 w = - \frac{M_o}{N(1 + \nu)} ,$$

and we may take:

$$M_o = K \log \rho, \tag{15.16}$$

where K is a proportionality constant. It follows by substitution that

$$\nabla^2 w = - \frac{K \log \rho}{N(1 + \nu)} . \tag{15.17}$$

Since the vertical shear, V_r, may be represented as:

$$V_r = - N \frac{\partial}{\partial \rho} \nabla^2 w = \frac{K}{\rho(1 + \nu)} , \tag{15.18}$$

it is easy to integrate the shear along a circle about the load and equate it to the load:

$$-P = \int_o^{2\pi} \frac{K}{\rho(1 + \nu)} \rho d\lambda = \frac{2\pi K}{1 + \nu} . \tag{15.19}$$

Therefore:

$$K = - \frac{P(1 + \nu)}{2\pi} , \tag{15.20}$$

and

$$M_o = - \frac{P(1 + \nu)}{2\pi} \log \rho . \tag{15.21}$$

Expression (15.21) will be seen to be identical with (13.9).

The transformation which maps the infinite sector into a unit circle with the load point, (r_1, θ_1), going into the origin may be written:

$$\zeta_1 = \frac{z_1^{\,k} - z^k}{z^k - \bar{z}_1^{\,k}} , \tag{15.22}$$

and it is easily shown that:

$$\rho_1 = \left\{ \frac{R^k + R^{-k} - 2\cos k(\theta_1 - \theta)}{R^k + R^{-k} - 2\cos k(\theta_1 + \theta)} \right\}^{1/2} . \tag{15.23}$$

It may be demonstrated that:

$$M_o = - \frac{P(1 + \nu)}{2\pi} \log \rho_1 , \tag{15.24}$$

satisfies the plate equation, satisfies the boundary conditions, satisfies the load discontinuity, and is identical with (14.5). Moreover one may show that equations (15.22) and (15.23) are limiting cases of (15.6) and (15.14) when in the latter equations r_2 becomes infinite.

(16) Evaluation of Twisting Moment in the Infinite Sector-Shaped Plate.

Take the point load on the axis of symmetry thus making $\theta_1 = \alpha/2$, then (14.5) becomes:

$$M_o = \frac{P(1 + \nu)}{4\pi} \log \frac{R^k + R^{-k} + 2\sin k\theta}{R^k + R^{-k} - 2\sin k\theta} . \tag{16.1}$$

It is evident from (14.8) and (16.1) that:

$$M_{r\theta} = Wk\cos k\theta\left(\frac{R^2 - 1}{R^2}\right)\left(\frac{R^k + R^{-k}}{(R^k + R^{-k})^2 - 4\sin^2 k\theta}\right), \quad (16.2)$$

where

$$W = P(1 - \nu)/4\pi.$$

If the twisting moment is to be evaluated along the edge of the sector where $\theta = \alpha$, then (16.2) reduces to:

$$M_{r\theta} = Wk\left(\frac{R^{k-2} - R^k}{R^{2k} + 1}\right). \quad (16.3)$$

Graph No. 4 illustrates the function (16.3) for various values of k which determine the size of the sectorial angle. It is noteworthy that for all angles less than 90°, the twisting moment at r = 0 vanishes, while for all angles greater than 90° the twisting moment is infinite at r = 0.

Summary

This thesis has been devoted to the solution of the problem of finding functions from which one may calculate the stresses and the deformations in certain thin elastic plates. The plates considered were of two types, a skew plate or a plate in the form of a parallelogram, and a sector-shaped plate. The method of solution followed the so-called Kirchoff elementary theory which disregards those internal stresses in planes normal to the plate. The assumptions are made that there is no compressive stress between horizontal layers of the plate, that there is no stretching of the middle surface and that plane sections normal to the middle surface before deformation remain plane and normal to that surface after bending. It is also assumed that the material is isotropic, that the stresses are below the proportional limit, and that the allowable deflections must not exceed one half the thickness of the plate. Although these assumptions might at first seem to be assumptions of convenience which allow the mathematical solution to be simplified, it is a fact that in the elastic range of the material and for small deflections, the Kirchoff method yields very reliable results. Precisely, this method involves the solution of the fourth order partial differential equation

$$N\nabla^4 w = p$$

where N is the so-called flexural coefficient of stiffness,

p is a function depending on the load and w is the deflection function.

The skew plate which was studied was of a particular shape that might be described as the combination of two isosceles right triangles placed in such a way that one pair of legs coincide. The first chapter is concerned with this skew plate supported on all edges so that each edge is free to rotate but is restrained from any deflection. The load function, p, is taken as a constant over the entire plate. In the second chapter the same skew plate is considered and the same edge conditions are imposed, but the load function is taken so that one has only a point load at the center of symmetry. Both of these chapters involve the construction of deflection functions which satisfy the plate equation as well as all of the boundary conditions. The particular way in which this was done was to construct two deflection functions for the plate, each function being valid in one of the above right triangles. It was then necessary to match these functions across the short diagonal where the two triangles joined and this process led to the necessity of solving two doubly infinite systems of equations involving infinitely many unknowns. The coefficients of the first five terms of the infinite series which represents the deflection function were calculated and graphs were drawn showing deflection curves along both diagonals. Twisting moments were also calculated along the edges and graphs were drawn.

The third chapter is a study of a thin plate in the form of a circular sector, and loaded with a point load at some point (r_1, θ_1). The solution is found in terms of infinite series for four distinct cases: (1) pinned edges all around, (2) pinned radial edges and clamped circular edge, (3) pinned radial edges and free circular edge, (4) Navier edge condition all around. The solution is also found for an infinite sector-shaped plate with a point load and in this case the moment sum, the bending moments, and the twisting moment are represented in closed form.

In the case of a Navier edge condition and in the case of an infinite sector it was found possible to arrive at the closed form for the moment sum by conformal mapping and the use of a Green's Function.

The deflection functions which have been found for the sector-shaped plate allow one to investigate the stresses and deformations near a corner which may vary from 0° to 180°. It should be noted that the curves obtained for the twisting moment along the edge of the skew plate near the 45° and 135° corners compare very favorably with the curves representing the twisting moment in the sector-shaped plate for the same angles.

LITERATURE CITED

1. Galerkin, B. G. _Uprugie Tonkie Plity_, (_Elastic Thin Plates_, translated title). pp. 225-262. Gosstroiuzdat. Leningrad, Russia. 1933.

2. Brigatti, Cecilia Vittoria. Applicazione del Metodo di H. Marcus al Calcolo della Piastra Parallelogrammico. _Ricerche di Ingegneria_. pp. 42-47. Vol. 6. March-April, 1938.

3. Jensen, Vernon P. Analysis of Skew Slabs. _University of Illinois Eng. Exp. Sta. Bul._ 332: p.5. 1941.

4. Anzelius, Adolf. Über die Elastische Deformation Parallelogrammformiger Platten. _Der Bauingenieur_. p. 478. Vol. 20. No. 35-36. Sept. 1939.

5. Vogt, Helmut. Die Berechnung Schiefwinkliger Platten und Plattenartiger Brückensysteme. _Beton un Eisen_. pp. 243-245. Vol. 39. No. 17. Sept. 1940.

6. Jensen, Vernon P. Analysis of skew Slabs. _University of Illinois Eng. Exp. Sta. Bul._ 332. 1941.

7. Nádai, A. _Die Elastischen Platten_. pp. 175-185. Julius Springer. Berlin, Germany. 1925.

8. Todhunter, Isaac, and Pearson, Karl. A History of the Theory of Elasticity and of the Strength of Materials. pp. 42; part 2; Vol. 2. The University Press. Cambridge, England. 1886.

9. Timoshenko, Stephen. Theory of Elasticity. p. 256. McGraw-Hill Book Co. New York. 1934.

10. Davis, Harold T. The Theory of Linear Operators. pp. 131-137. Principia Press. Bloomington, Indiana. 1936.

11. Davis, Harold T. The Theory of Linear Operators. pp. 130-131. Principia Press. Bloomington, Indiana. 1936.

12. Holl, D. L. and Murphy, Glenn. Special research project on skew and sector-shaped plates sponsored by Sigma Xi. Private communication. 1942.

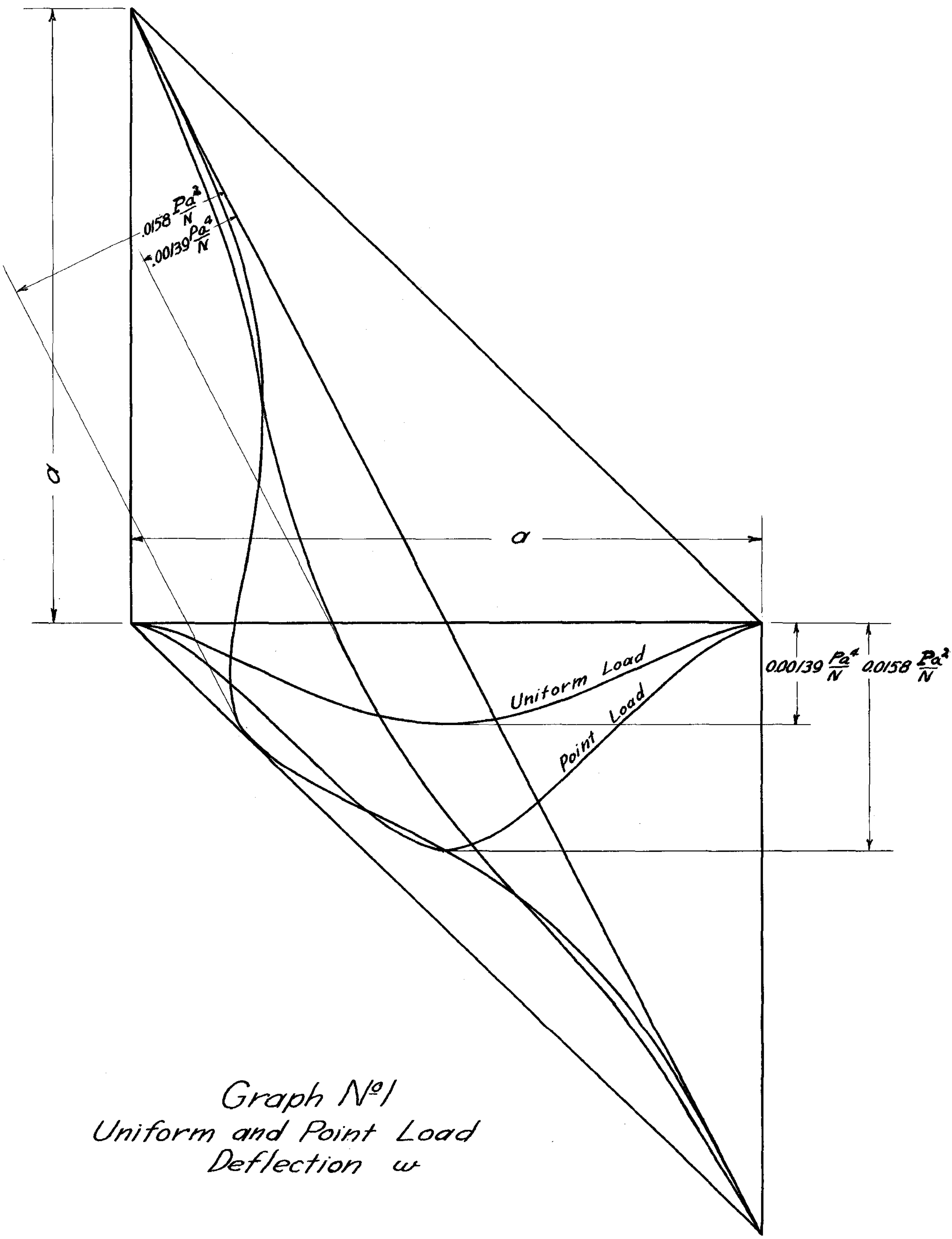
13. Galerkin, B. G. Uprugie Tonkie Plity, (Elastic Thin Plates, translated title). pp. 262-325. Gosstroĭuzdat. Leningrad, Russia. 1933.

14. Loney, S. L. Plane Trigonometry. Pt. 2, Analytical Trigonometry. pp. 119-120. Cambridge University Press. Cambridge, England. 1925.

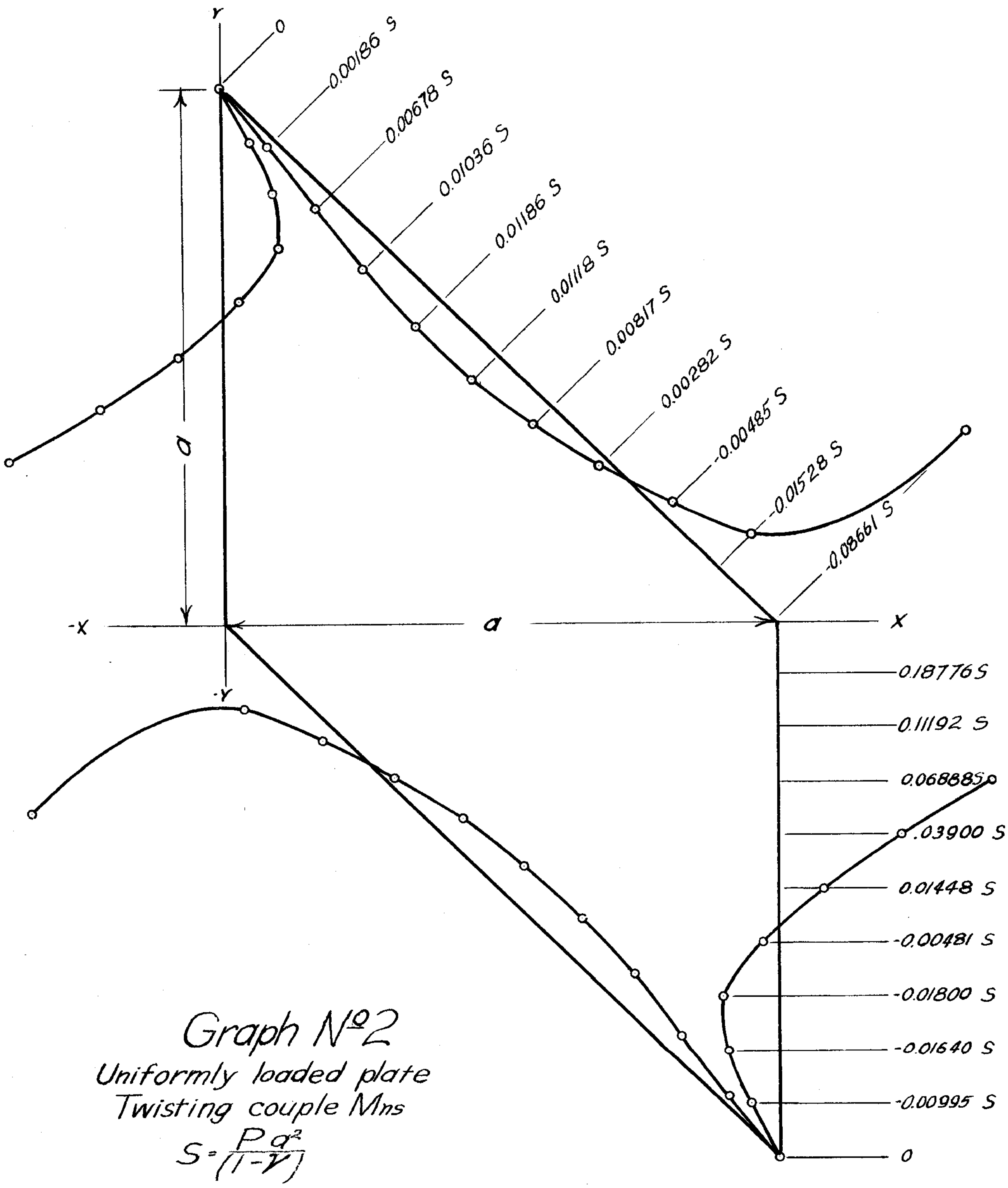
15. Newmark, N. M. Moments near the corner of a sector-shaped

slab. Private communication. 1940.

16. Bateman, H. Partial Differential Equations of Mathematical Physics. pp. 292-294. The MacMillan Co. 1932.

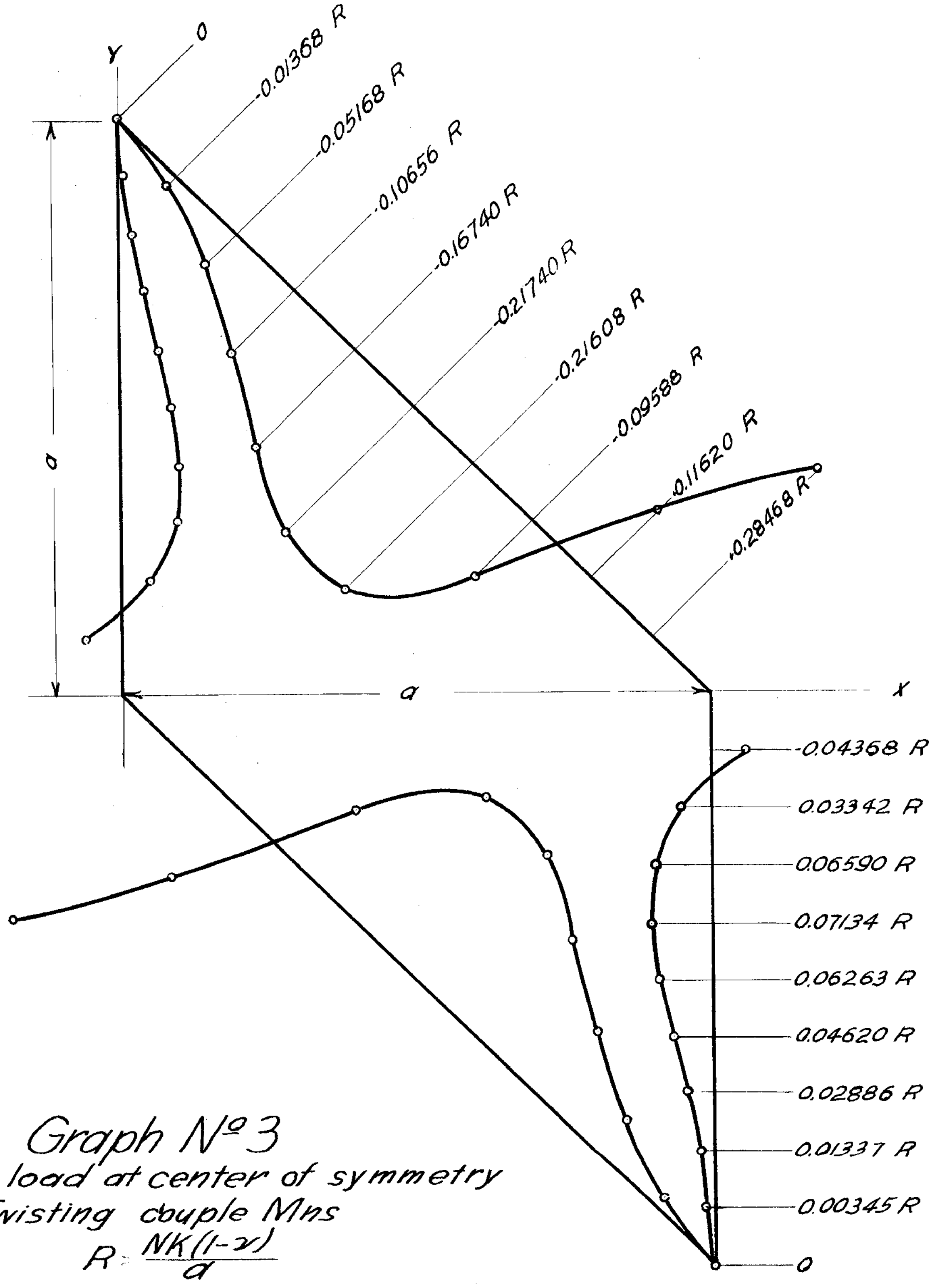


Graph No 1
Uniform and Point Load
Deflection w

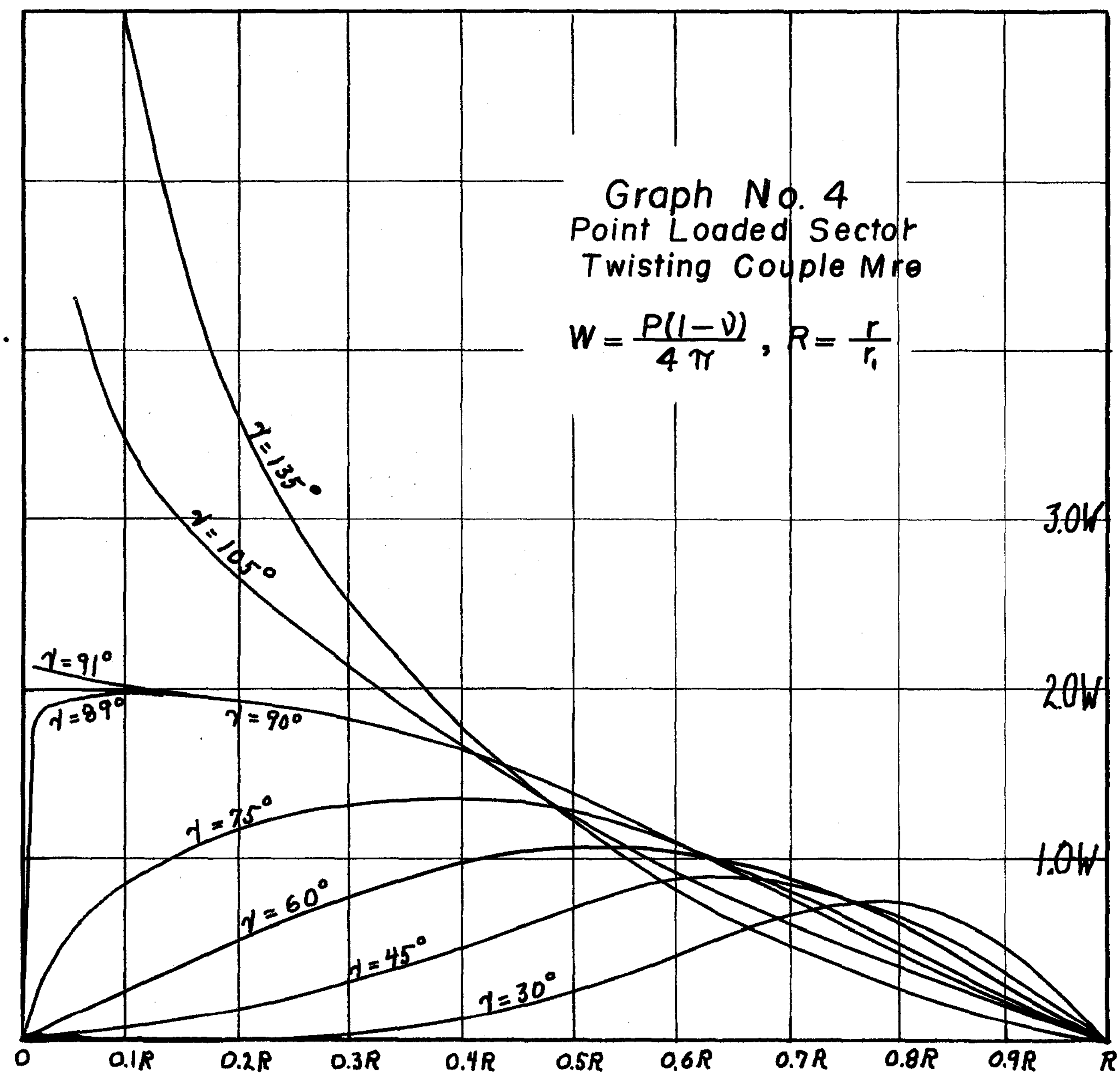


Graph №2
Uniformly loaded plate
Twisting couple M_{ns}

$$S = \frac{Pa^2}{(1-\nu)}$$



Graph Nº 3
Point load at center of symmetry
Twisting couple M_{ns}
$R = \frac{NK(1-\nu)}{a}$


Graph No. 4
Point Loaded Sector
Twisting Couple Mre
$W = \frac{P(1-\nu)}{4\pi}$, $R = \frac{r}{r_1}$
$\gamma = 135°$
$\gamma = 105°$
$\gamma = 91°$
$\gamma = 89°$
$\gamma = 90°$
$\gamma = 75°$
$\gamma = 60°$
$\gamma = 45°$
$\gamma = 30°$
3.0W
2.0W
1.0W
0
0.1R
0.2R
0.3R
0.4R
0.5R
0.6R
0.7R
0.8R
0.9R
R